Adding Level-1 Predictors to Multilevel Models for Clustered Data

- Topics:
  - Fixed slopes of level-1 person predictors
    - Cluster-mean-centering, constant-centering, and latent centering
  - Random slopes of level-1 person predictors
  - Cross-level interactions and systematically varying effects

- By Lesa Hoffman, Professor of Educational Measurement and Statistics in the University of Iowa College of Education
  - Presented March 16, 2023, as part of the APA Free Science Trainings Series
  - Btw, my full course at Univ of Iowa on Clustered Multilevel Models is in Fall 2023!

Clustered Data MLMs Part 1: Review

- Multilevel models (MLMs) are used to quantify and predict how much of an outcome's total variation is due to each dimension of sampling
- Empty means, two-level model for level-1 person $\boldsymbol{p}$ in level-2 cluster $\boldsymbol{c}$:

Level-1: $y_{pc} = \beta_{0c} + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + U_{0c}$

γ_{00} = fixed intercept (mean of cluster means)
U_{0c} = level-2 random intercept (with variance $\tau^2_{U_0}$)
e_{pc} = level-1 residual (with variance σ^2_e)

- **Total** outcome variation is partitioned into **two uncorrelated sources**:
  - **Level-2 between**-cluster (BC) mean differences → random intercept $\tau^2_{U_0}$
  - **Level-1 within**-cluster (WC) cluster differences → residual σ^2_e
  - Dependency effect size via Intraclass Correlation: $\mathbf{ICC} = \tau^2_{U_0} / (\tau^2_{U_0} + \sigma^2_e)$
    - ICC = proportion of total variance due to cluster mean differences
    - ICC = average correlation of persons from same cluster
- Fixed slopes of level-2 predictors explain cluster mean differences, thereby reducing the level-2 random intercept variance $\tau^2_{U_0}$

Level-1 Predictors: What **Not** to Do!

- Level-2 predictors ($\boldsymbol{L2x_c}$ below) are **cluster** characteristics
- Level-1 predictors ($\boldsymbol{L1x_{pc}}$ below) are **person** characteristics
  - *What if we added a L1 predictor directly (as we did before at L2)?*

Level-1: $\boldsymbol{y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}}$

Level-2: $\boldsymbol{\beta_{0c} = \gamma_{00} + \gamma_{01}(L2x_c) + U_{0c}}$

$\boldsymbol{\beta_{1c} = \gamma_{10}}$

$\boldsymbol{\gamma_{00}}$	= fixed intercept (at pred=0)
$\boldsymbol{\gamma_{01}}$	= fixed slope of $L2x_c$
$\boldsymbol{\gamma_{10}}$	= fixed slope of $L1x_{pc}$
$\boldsymbol{U_{0c}}$	= level-2 random intercept
$\boldsymbol{e_{pc}}$	= level-1 residual

  - First subscript = which beta in level-1 model
    Second subscript = order of predictor in level-2 model

- All good, right? Many researchers mistakenly think so, but this model is **VERY LIKELY to be mis-specified**…
  - … For the **exact same reasons** we need MLM in the first place!

Level-1 (Person-Level) Predictors

- Modeling level-1 predictors is complicated (and often done incorrectly) because **each level-1 predictor is usually really 2 predictor variables** (each with their own slope), **not 1**

- Textbook example: Student Socioeconomic Status (SES)
  - Some **kids** have higher SES than others in their school:
    - **L1 WC variation in SES** *(represented directly as deviation from school mean)*
  - Some **schools** have more high-SES students than other schools:
    - **L2 BC variation in SES** *(represented as school mean or via external info)*

- Can quantify each source of variance with an empty model ICC
  - ICC = (L2 between variance) / (L2 between variance + L1 within variance)
  - **ICC < 1?** L1 predictor has **WC** variation (so it *could* have a **L1 WC** slope)
  - **ICC > 0?** L1 predictor has **BC** variation (so it *could* have a **L2 BC** slope)

Between- vs. Within-Cluster Effects

- Between- and within-cluster slopes in SAME direction
  - SES → Achievement in students
    - **WC: Kids with more money than other kids in their school may have greater achievement than other kids in their school (regardless of school mean SES)**
    - **BC: Schools with more money than other schools may have greater mean achievement than schools with less money**
- Between- and within-cluster slopes in OPPOSITE directions
  - Body mass → life expectancy in animals (Curran and Bauer, 2011)
    - **WC: Within a species, relatively bigger animals have shorter life expectancy (e.g., over-weight ducks die sooner than healthy-weight ducks)**
    - **BC: Larger species tend to have longer life expectancies than smaller species (e.g., whales live longer than cows, cows live longer than ducks)**
- L1 within-cluster and L2 between-cluster slopes usually differ
  - Why? Because variables have different **meanings** at each level!
  - Why? Because variables have different **scales** at each level!

What **Not** to Do: Smushed Effects!

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + U_{0c}$

$\beta_{1c} = \gamma_{10}$

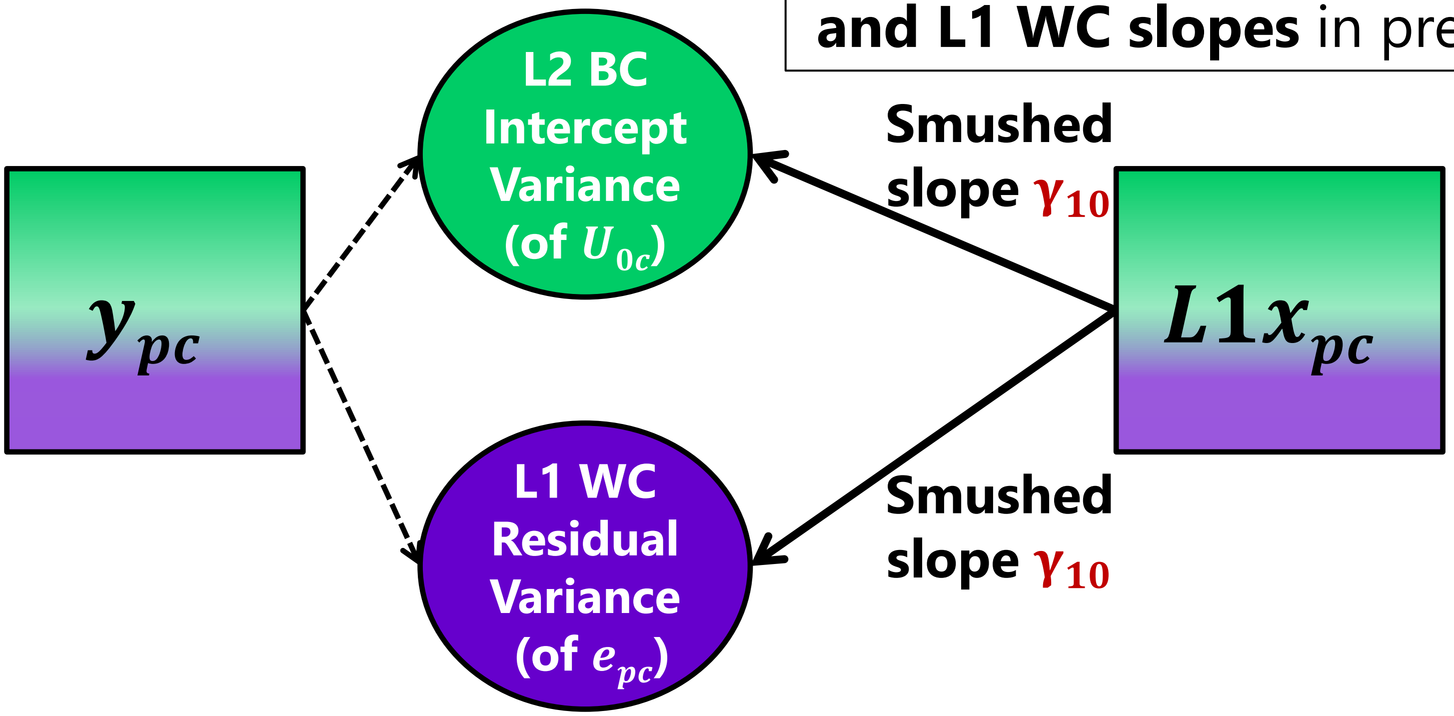
γ_{10} **= smushed effect** (see also *conflated*, *convergence*, or *composite* effect) that assumes equal within- and between-cluster slopes

- If level-1 predictor has both level-2 between and level-1 within variation, then its **one fixed slope has to do the work of two predictors**!
- A **smushed effect** is a **weighted combination of the L1 within and L2 between slopes,** usually **closer to the L1 within slope** (due to larger $L1n$), and thus the L2 between model will be more affected by smushing
- Btw, **smushing** is seen in econometrics (aka, "**endogeneity**" problem) in the context of when to model cluster dependency using fixed effects (i.e., turn cluster ID into a categorical predictor) instead of a random intercept
  - A **smushed effect creates correlation** between the L1 predictor and the L2 random intercept (because the **predictor's L2 effect is modeled wrong**)
  - Smushing is solved when using **fixed effects for cluster ID,** such that the L2 effect of the L1 predictor is then controlled for in "common" variance
  - But we can still avoid smushed effects when using a cluster random intercept…. Next are the 3 main ways to do so!

Univariate MLM: Adding a Level-1 Predictor Without Addressing Level-2 Part = Smushing

BC and WC variance in the **observed level-1 y_{pc} outcome** is partitioned by the **model** into estimated **variance components**

Observed level-1 $L1x_{pc}$ predictor still has both BC and WC variance. AND given that $L1x_{pc}$ has only **one fixed slope**, it captures a smushed effect that presumes **equal L2 BC and L1 WC slopes** in predicting y_{pc}!



3 Kinds of Fixed Slopes for L1 Predictors

- **Is there a Level-1 Within-Cluster (WC) slope?**
  - If you have a higher $\boldsymbol{L1x_{pc}}$ predictor value *than others in your cluster*, do you also have a higher (or lower) $\boldsymbol{y_{pc}}$ outcome value *than others in your cluster*?
  - If so, the **level-1 within-cluster *part* of the L1 predictor** will reduce the level-1 residual variance ($\boldsymbol{\sigma_e^2}$) of the $\boldsymbol{y_{pc}}$ outcome

- **Is there a Level-2 Between-Cluster (BC) slope?**
  - Do clusters with higher average $\boldsymbol{L1x_{pc}}$ predictor values *than other clusters* also have higher (or lower) average $\boldsymbol{y_{pc}}$ outcomes *than other clusters*?
  - If so, the **level-2 between-cluster *part* of the L1 predictor** will reduce level-2 random intercept variance ($\boldsymbol{\tau_{U_0}^2}$) of the $\boldsymbol{y_{pc}}$ outcome

- **Is there a Level-2 Contextual slope: Do the L2 BC and L1 WC slopes differ?**
  - After controlling for the actual value of L1 predictor, is there still **an incremental contribution** from the **level-2 between-cluster *part* of the L1 predictor** (i.e., does a cluster's general tendency matter beyond a person's $\boldsymbol{L1x_{pc}}$ value)?
  - Equivalently, the **Level-2 Contextual slope** = **L2 BC slope** − **L1 WC slope**, so the Level-2 Contextual slope directly tests **if a smushed slope is ok (pry not!)**

3 Options to Prevent Smushed Slopes

- Within Univariate MLM framework (predict only one outcome):
  1. **Cluster-mean-centering**: manually carve up L1 predictor into its level-specific parts using observed variables (1 predictor per level)
     - More generally, this is "**variable-centering**" because you are **subtracting a variable** (e.g., the cluster mean here; could use other cluster variables)
     - Will always yield **level-1 within slopes** and **level-2 between slopes**!
  2. **Grand-mean-centering**: do NOT carve up L1 predictor into its level-specific parts, but add level-2 mean to distinguish level-specific slopes
     - More generally, this is "**constant-centering**" because you are **subtracting a constant** while still keeping all levels of variance in the L1 predictor
     - **Choice of constant is irrelevant** (changes where 0 is, not what variance it has)
     - Will always yield **level-1 within slopes** and **level-2 contextual slopes**!
- Within Multivariate MLM framework (i.e., via Multilevel-SEM):
  3. **Latent-centering**: Treat the L1 predictor as another outcome → let the model carve it up into **level-specific latent variables**
     - Best in theory, but the type of level-2 slope (between or contextual) depends on model type, syntax type, and the estimator in Mplus! (Hoffman, 2019)

Option 1. Cluster-Mean-Centering (C-MC)

- We partition the L1 predictor $\boldsymbol{L1x_{pc}}$ into two variables that directly represent its **L2 between**-cluster (BC) and **L1 within**-cluster (WC) sources of variation, and **include these variables as the predictors:**

- **Level-2 Between predictor = cluster mean of $L1x_{pc}$**
  - $\boldsymbol{CMx_c = \overline{L1x}_c - C_2}$
  - CMx_c is centered at constant C_2, chosen for meaningful 0 (e.g., sample mean)
  - CMx_c is positive? Above sample mean → "more than other clusters"
  - CMx_c is negative? Below sample mean → "less than other clusters"

- **Level-1 Within predictor = deviation from cluster mean of $L1x_{pc}$**
  - $\boldsymbol{WCx_{pc} = L1x_{pc} - \overline{L1x}_c}$ *(uncentered cluster mean $\boldsymbol{\overline{L1x}_c}$ is used)*
  - WCx_{pc} is NOT centered at a constant – **we subtract a VARIABLE instead**
  - WCx_{pc} is positive? Above your cluster mean → "more than my cluster"
  - WCx_{pc} is negative? Below your cluster mean → "less than my cluster"

Cluster-MC L1 Predictor + Cluster Mean → WC and BC effects directly through **separate** parameters

$L1x_{pc}$ **is cluster-mean-centered into** WCx_{pc}**, with** CMx_c **at L2:**

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(WCx_{pc}) + e_{pc}$

$WCx_{pc} = L1x_{pc} - \overline{L1x}_c$ → **only has L1 within variation**

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

$CMx_c = \overline{L1x}_c - C_2$ → **only has L2 between variation**

γ_{10} **= within effect of having more** $L1x_{pc}$ ***than others in your cluster***

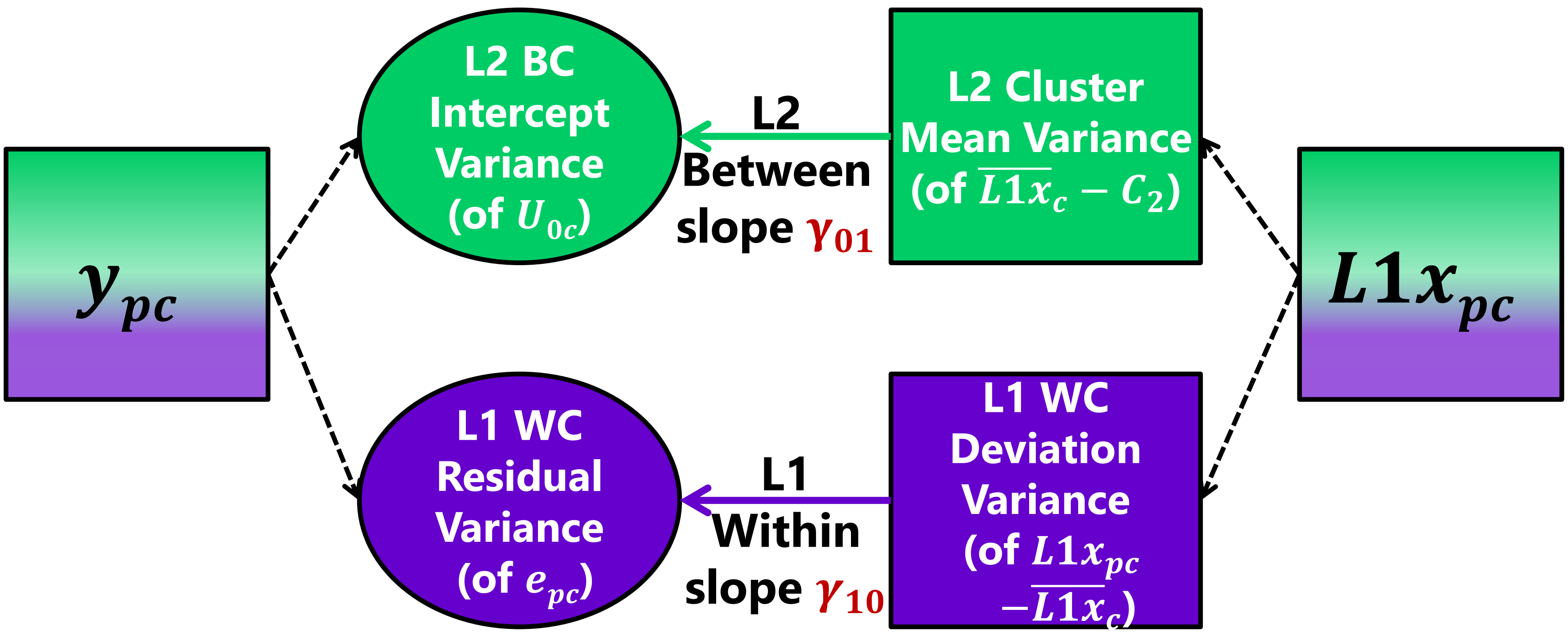
γ_{01} **= between effect of having more** $\overline{L1x}_c$ ***than other clusters***

Because WCx_{pc} **and** CMx_c **are uncorrelated, each gets the total effect for its level (L1 = within, L2 = between)**

Univariate MLM: Cluster-Mean-Centering

Model-based partitioning of level-1 y_{pc} outcome into level-specific **latent variables**

Manual partitioning of level-1 $L1x_{pc}$ predictor into level-specific **observed variables**



Why not let the model make variance components for $L1x_{pc}$, too?
That is option 3, multivariate MLM (or "multilevel SEM"): stay tuned…

Adding L2 Between and L1 Within Predictors: (2a) Syntax by Univariate MLM Program

SAS:

```
PROC MIXED DATA=work.Example COVTEST NOCLPRINT IC METHOD=REML;
     CLASS schoolID;
     MODEL langpost = hw2 mixgrd CMverb10 WCverb / SOLUTION DDFM=Satterthwaite;
     RANDOM INTERCEPT / TYPE=UN SUBJECT=schoolID;
     ESTIMATE "L2 Contextual Effect of Verbal"  CMverb10 1 WCverb -1;
RUN;
```

R lmer from lme4 package—using lmerTest package to get Satterthwaite denominator DF and contest1D:

```
name = lmer(data=Example, REML=TRUE,
            formula=langpost~1+hw2+mixgrd+CMverb10+WCverb+(1+|schoolID))
summary(name, ddf="Satterthwaite")
contest1D(name, ddf="Satterthwaite", L=c(0,0,0,1,-1)) # L2 Contextual effect of verbal
```

STATA:

```
mixed langpost c.hw2 c.mixgrd c.CMverb10 c.WCverb, || schoolID:, ///
      reml dfmethod(satterthwaite) dftable(pvalue) nolog
lincom c.CMverb10*1 + c.WCverb*-1, small // L2 Contextual effect of verbal
```

SPSS:

```
MIXED langpost BY schoolID WITH hw2 mixgrd CMverb10 WCverb
      /METHOD    = REML
      /CRITERIA  = DFMETHOD(SATTERTHWAITE)
      /PRINT     = SOLUTION TESTCOV
      /FIXED     = hw2 mixgrd CMverb10 WCverb
      /RANDOM    = INTERCEPT | COVTYPE(UN) SUBJECT(schoolID)
      /TEST      = "L2 Contextual effect of verbal" CMverb10 1 WCverb -1.
```

Example: Cluster-MC Level-1 Predictor

Example from Snijders & Bosker (2012) ch. 9: Predicting language outcomes for 3,566 students (p) from 191 schools (c) → **adding student verbal ability**

Level-1: $Lang_{pc} = \beta_{0c} + \beta_{1c}(Verbal_{pc} - \overline{Verbal}_c) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(HW_c - 2) + \gamma_{02}(MixGrd_c) + \gamma_{03}(\overline{Verbal}_c - 10) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

Results from SAS MIXED:

L1 WCverb = $Verbal_{pc} - \overline{Verbal}_c$
L2 CMverb10 = $\overline{Verbal}_c - 10$

Solution for Fixed Effects					
Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	41.5794	0.3624	172	114.73	<.0001
hw2	-0.05255	0.4585	179	-0.11	0.9089
mixgrd	-1.1209	0.5157	197	-2.17	0.0309
CMverb10	3.6599	0.2709	207	13.51	<.0001
WCverb	2.4227	0.05718	3373	42.37	<.0001

Btw, L2 Contextual = 1.237, SE = 0.277, p < .0001

Covariance Parameter Estimates					
Cov Parm	Subject	Estimate	Standard Error	[illegible]	[illegible]
UN(1,1)	schoolID	8.3939	1.1326	[illegible]	[illegible]
Residual		40.5508	0.9875	[illegible]	[illegible]

From empty model to compare:

Covariance Parameter Estimates					
Cov Parm	Subject	Estimate	Standard Error	[illegible]	[illegible]
UN(1,1)	schoolID	17.8085	2.3063	[illegible]	[illegible]
Residual		62.2296	1.5179	[illegible]	[illegible]

Example: Cluster-MC Level-1 Predictor

Model for the Means (relevant parameters only):

- γ_{00} = **41.58** = fixed **intercept**: expected language for students in a school with homework=2 (~mean), mixgrd=0 (=not mixed), and school mean verbal = 10; for a student whose verbal = 10
- γ_{03} = **3.66*** = fixed **BC slope** of **school verbal**: difference in **school mean** language per unit higher mean verbal ability *than other schools*
- γ_{10} = **2.42*** = fixed **WC slope** of **student verbal**: difference in **student** language per unit higher verbal ability *than their school mean*

Model for the Variance:

- $\boldsymbol{U_{0c}}$ = level-2 random intercept = deviation between actual and predicted school mean language for school c (with variance $\boldsymbol{\tau^2_{U_0}}$ = **8.39**)
  - Pseudo-$R^2_{U0} = \frac{\mathbf{17.809-8.394}}{\mathbf{17.809}}$ **= .529 → 52.9%** explained (of original 22.3% L2 BC)
- $\boldsymbol{e_{pc}}$ = level-1 residual = deviation of the actual outcome for student p from their outcome predicted by $\boldsymbol{\beta_{0c}}$ and $\boldsymbol{\beta_{1c}}$ (with variance $\boldsymbol{\sigma^2_e}$ = **40.55**)
  - Pseudo-$R^2_e = \frac{\mathbf{62.230-40.551}}{\mathbf{62.230}}$ **= .348 → 34.8%** explained (of original 77.7% L1 WC)

3 Kinds of Fixed Slopes for L1 Predictors

- **2 kinds of slopes Cluster-Mean-Centering tells us *directly*:**

- **Is there a Level-1 Within-Cluster (WC) slope?**
  - If you have higher predictor values than the rest of your cluster, do you also have higher outcomes values than the rest of your cluster, such that the within-cluster deviation of the L1 predictor accounts for L1 residual outcome variance (σ_e^2)?
  - **Given directly by fixed slope of WCx_{pc} regardless of whether CMx_c is there**
  - Note: L1 slope multiplies the **relative** value of $L1x_{pc}$, NOT the **original** $L1x_{pc}$

- **Is there a Level-2 Between-Cluster (BC) slope?**
  - Do clusters with higher predictor values than other clusters (*on average*) also have higher outcomes than other clusters (*on average*), such that the cluster mean of the L1 predictor accounts for level-2 random intercept variance ($\tau_{U_0}^2$)?
  - **Given directly by fixed slope of CMx_c regardless of whether WCx_{pc} is there**
  - Note: BC slope is NOT controlling for the original $L1x_{pc}$ for each person

3rd Kind of Slope for L1 Predictors

- **What Cluster-Mean-Centering DOES NOT tell us *directly*:**

- **Is there a Level-2 Contextual effect: Do the BC and WC slopes differ?**
  - After controlling for the original value of the L1 predictor per person, is there still **an incremental contribution from having a higher cluster mean** of the L1 predictor (i.e., does a cluster's general tendency for the predictor explain more $\tau^2_{U_0}$ above and beyond just the person-specific value of the L1 predictor)?
  - If there is no contextual effect, then the L1 predictor's **L2 BC** and **L1 WC** slopes show **convergence**, which means their effects are of equivalent magnitude

- **To answer this question about the Level-2 Contextual effect for the incremental contribution of the cluster mean, we have two options:**
  - Still use Cluster-MC, and ask for the **contextual slope = between − within** (via SAS ESTIMATE, R contest1D, SPSS TEST, STATA LINCOM, Mplus NEW…)
  - Use "**constant-centering**" for the L1 predictor: $L1x_{pc} = L1x_{pc} - C_1$
    → **centered at CONSTANT C_1, NOT A LEVEL-2 VARIABLE**
    - Which constant only matters for the reference point; it could be the grand mean or any (even 0)

Why the Difference in the Level-2 Effect? Remember Regular Old Regression…

- In this model: $y_p = \beta_0 + \beta_1(x1_p) + \beta_2(x2_p) + e_p$
- If $x1_p$ and $x2_p$ **ARE NOT** correlated:
  - β_1 carries **ALL the relationship** between $x1_p$ and y_p
  - β_2 carries **ALL the relationship** between $x2_p$ and y_p

- If $x1_p$ and $x2_p$ **ARE** correlated:
  - β_1 is **different than** the bivariate relationship between $x1_i$ and y_i
    - "Unique" effect of $x1_p$ *controlling for* $x2_p$ (i.e., *holding* $x2_p$ *constant)*
  - β_2 is **different than** the bivariate relationship between $x2_i$ and y_i
    - "Unique" effect of $x2_p$ *controlling for* $x1_p$ (i.e., *holding* $x1_p$ *constant)*

- **Hang onto that idea…**

Cluster-Mean-Centering vs. Constant-Centering for Level-1 Predictors

Level 2		Original	Cluster-MC Level 1	Grand-MC Level 1
$\overline{L1x_c}$	$CMx_c = \overline{L1x_c} - 5$	$L1x_{pc}$	$WCx_{pc} = L1x_{pc} - \overline{L1x_c}$	$L1x_{pc} = L1x_{pc} - 5$
3	−2	2	−1	−3
3	−2	4	1	−1
7	2	6	−1	1
7	2	8	1	3

Same L2 CMx_c goes into the model given either way of centering the L1 predictor $L1x_{pc}$

In **variable-centering** (C-MC), the level-2 BC mean variation is gone from WCx_{pc}, so it is NOT CORRELATED with CMx_c

In **constant-centering**, the level-2 BC mean variation is still inside $L1x_{pc}$, so it IS STILL CORRELATED with CMx_c

So the effects of CMx_c and $L1x_{pc}$ when included together under constant-centering will be different than if either predictor were included by itself…

Level-1 Predictor + Cluster Mean

→ Model tests difference of WC vs. BC effects

$L1x_{pc}$ is constant-centered, but WITH CMx_c at Level 2:

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

$L1x_{pc} = L1x_{pc} - C_1$ → still has both L2 between and L1 within variation

$CMx_c = \overline{L1x}_c - C_2$ → only has L2 between variation

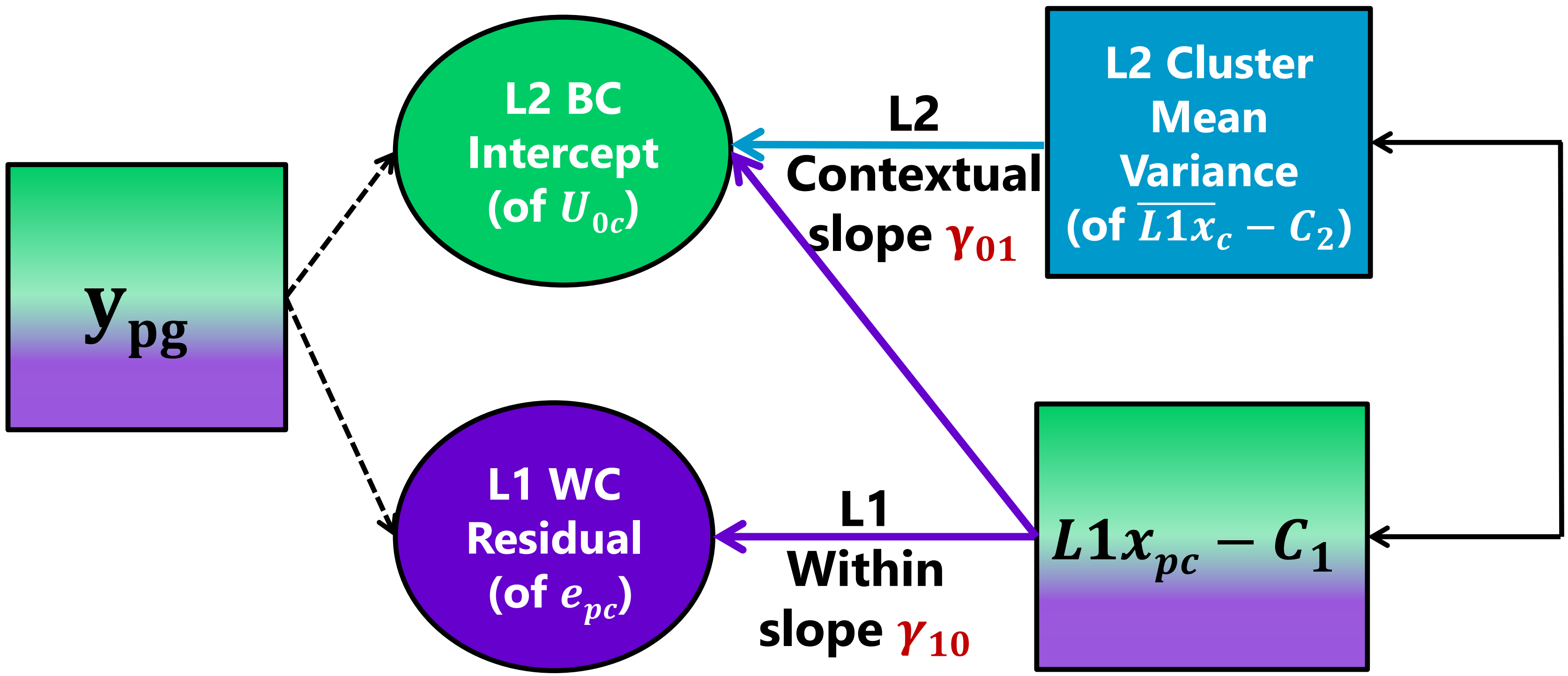
γ_{01} becomes the within effect → *unique* L1 effect after controlling for L2 CMx_c

γ_{01} *becomes* the L2 Contextual slope that indicates <u>how the L2 BC effect differs from the L1 WC effect</u>
→ *unique* level-2 slope after controlling for $L1x_{pc}$
→ does cluster mean matter beyond person value?

Constant-Centering + Cluster Mean

Model-based partitioning of y_{pc} outcome into level-specific **latent variables**

$L1x_{pc}$ **is still NOT partitioned**, but cluster mean $\overline{L1x_c} - C_2$ is added to allow an **incremental L2 effect**



L2 BC slope = L1 WC slope + Level-2 Contextual slope

Because original $L1x_{pc}$ still has L2 BC variance, it still carries ***some*** of the L2 BC effect…

Adding L2 Contextual and L1 Within Predictors: (3a) Syntax by Univariate MLM Program

SAS:

```
PROC MIXED DATA=work.Example COVTEST NOCLPRINT IC METHOD=REML;
     CLASS schoolID;
     MODEL langpost = hw2 mixgrd CMverb10 verb10 / SOLUTION DDFM=Satterthwaite;
     RANDOM INTERCEPT / TYPE=UN SUBJECT=schoolID;
     ESTIMATE "L2 Between Effect of Verbal"  CMverb10 1 verb10 1;
RUN;
```

R lmer from lme4 package—using lmerTest package to get Satterthwaite denominator DF and contest1D:

```
name = lmer(data=Example, REML=TRUE,
            formula=langpost~1+hw2+mixgrd+CMverb10+verb10+(1+|schoolID))
summary(name, ddf="Satterthwaite")
contest1D(name, ddf="Satterthwaite", L=c(0,0,0,1,1)) # L2 Between effect of verbal
```

STATA:

```
mixed langpost c.hw2 c.mixgrd c.CMverb10 c.verb10, || schoolID:, ///
      reml dfmethod(satterthwaite) dftable(pvalue) nolog
lincom c.CMverb10*1 + c.verb10*1, small // L2 Between effect of verbal
```

SPSS:

```
MIXED langpost BY schoolID WITH hw2 mixgrd CMverb10 verb10
      /METHOD   = REML
      /CRITERIA = DFMETHOD(SATTERTHWAITE)
      /PRINT    = SOLUTION TESTCOV
      /FIXED    = hw2 mixgrd CMverb10 verb10
      /RANDOM   = INTERCEPT | COVTYPE(UN) SUBJECT(schoolID)
      /TEST     = "L2 Between effect of verbal" CMverb10 1 verb10 1.
```

Example: Constant-C Level-1 Predictor

Level-1: $Lang_{pc} = \beta_{0c} + \beta_{1c}(Verbal_{pc} - 10) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(HW_c - 2) + \gamma_{02}(MixGrd_c) + \gamma_{03}(\overline{Verbal}_c - 10) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

Fixed Effects from SAS MIXED (model for variance is same):

Constant-C from above:
L1 verb10 = $Verbal_{pc} - 10$ (**differs**)
L2 CMverb10 = $\overline{Verbal}_c - 10$ (**same**)

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	41.5794	0.3624	172	114.73	<.0001
hw2	-0.05255	0.4585	179	-0.11	0.9089
mixgrd	-1.1209	0.5157	197	-2.17	0.0309
CMverb10	1.2372	0.2769	226	4.47	<.0001
verb10	2.4227	0.05718	3373	42.37	<.0001

← L2? →

L1 WC

Compared to Cluster-MC from before:
L1 WCverb = $Verbal_{pc} - \overline{Verbal}_c$ (**differs**)
L2 CMverb10 = $\overline{Verbal}_c - 10$ (**same**)

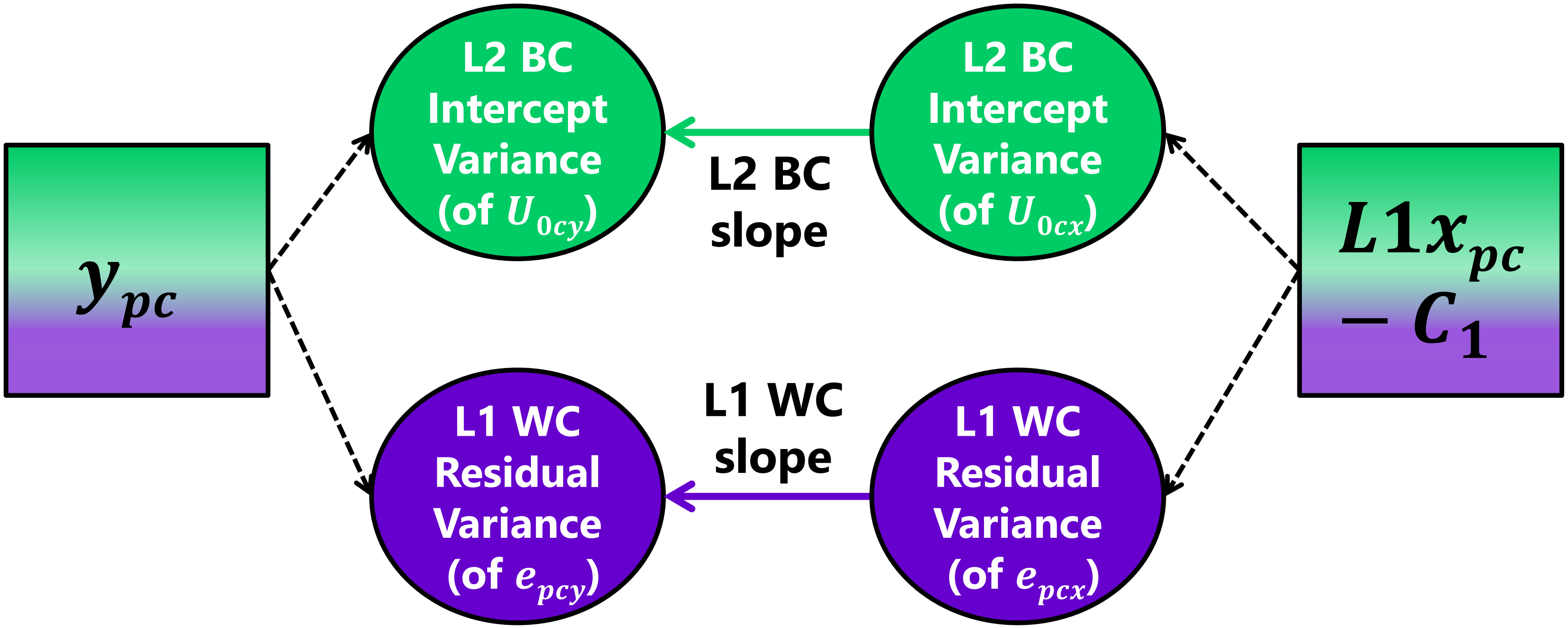
Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	41.5794	0.3624	172	114.73	<.0001
hw2	-0.05255	0.4585	179	-0.11	0.9089
mixgrd	-1.1209	0.5157	197	-2.17	0.0309
CMverb10	3.6599	0.2709	207	13.51	<.0001
WCverb	2.4227	0.05718	3373	42.37	<.0001

- **L2 Contextual** slope = **1.24** using constant-C L1 (= **Between** − **Within**)
- **L2 Between** slope = **3.66** using cluster-MC L1 (= **Contextual** + **Within**)
- The **smushed** slope would have been **2.472** = Within (close) = Between (too small)!

Option 3: Latent-Centering in Multivariate MLM

Model-based partitioning of level-1 y_{pc} outcome into level-specific **latent variables**

Model-based partitioning of level-1 $L1x_{pc}$ predictor (= outcome now) into level-specific **latent variables**



Univariate MLM software can be tricked into multivariate MLM if the relationships between X and Y at each level are phrased as covariances, but not if you want directed regressions (or moderators thereof)

Mplus M-SEM: Latent Centering of L1 Verbal

```
TITLE:  Model2a: Latent Centering of Student Verbal Ability Predicting Language
        Specifying L1 effect in WITHIN model directly
DATA:   FILE = ExampleData.csv;  ! Can just list file if syntax in same folder
        TYPE = INDIVIDUAL; FORMAT = FREE;   ! Defaults
VARIABLE:
! List of ALL variables in stacked data file, in order (up to 8 characters)
  NAMES = schoolID studID lang verbal homework mixgrd;
! List of ALL variables used in model (DEFINED variables go last)
  USEVARIABLES = lang mixgrd hw2 verb10;
! Missing data codes (here, -999)
  MISSING = ALL (-999);
! Identify level-2 ID
  CLUSTER = schoolID;
! Predictor variables with variation ONLY at level 1 -- none here
  WITHIN = ;
! Predictor variables with variation ONLY at level 2 (DEFINED last)
  BETWEEN = mixgrd hw2;

DEFINE:     hw2 = homework-2;        ! Center L2 homework at 2
            verb10 = verbal - 10;    ! Center L1 verbal at 10
ANALYSIS:   TYPE = TWOLEVEL RANDOM;  ! 2-level model with random slopes
            ESTIMATOR = ML;          ! Can also use MLR for non-normality
MODEL:
! Level-1, Within-Cluster (WC) Model
%WITHIN%
  lang;                         ! L1 residual variance in lang
  verb10;                       ! L1 residual variance in verbal (new)
  lang ON verb10 (within);      ! NO Placeholder, L1 Within verbal -> lang

! Level-2, Between-Cluster Model
%BETWEEN%
 [lang];                        ! Fixed intercept for lang
  lang;                         ! L2 random intercept variance in lang
 [verb10];                      ! Fixed intercept for verbal (new)
  verb10;                       ! L2 random intercept variance in verbal (new)
  lang ON hw2 mixgrd;           ! Between fixed slopes of L2 preds -> lang
  lang ON verb10 (between);     ! Between fixed slope of verbal -> lang

MODEL CONSTRAINT:               ! Linear combinations of fixed effects
 NEW(context);                  ! Name each new created fixed effect
 context = between - within;    ! L2 Contextual fixed slope of verbal -> lang
```

	Estimate	S.E.	P-Value
Within Level			
LANGPOST ON			
VERB10	2.425	0.057	0.000
Variances → NEW!			
VERB10	3.688	0.090	0.000
Residual Variances			
LANGPOST	40.536	0.987	0.000
Between Level			
LANGPOST ON			
HW2	-0.076	0.456	0.867
MIXGRD	-1.193	0.513	0.020
VERB10	4.239	0.421	0.000
Means → NEW!			
VERB10	-0.017	0.062	0.786
Intercepts			
LANGPOST	41.597	0.360	0.000
Variances → NEW!			
VERB10	0.510	0.080	0.000
Residual Variances			
LANGPOST	7.765	1.126	0.000
New/Additional Parameters			
CONTEXT	1.814	0.429	0.000

Relative to the cluster-MC univariate MLM (using REML estimation), in the latent-centered multivariate MLM (using ML estimation), the **L2 Between effect is larger** (4.24 vs. 3.66), a phenomenon known as "**Lüdtke's bias**"

I *Usually* Prefer Variable-Centering (using observed or latent variables)…

- **…because constant-centering is much easier to screw up! ☺**
- Table 1 below from: Hoffman, L., & Walters, R. W. (2022). Catching up on multilevel modeling. *Annual Review of Psychology, 73*, 629-658.

Table 1 Predictor effect type by model specification

Centering strategy for level-1 predictor (constant-centered level-2 predictor)	Fixed effect type by predictors included		
	Level-1 only	Level-2 only	Both levels
Variable-centered level-1			
Level-1 predictor: $L1x_{wb} = x_{wb} - \bar{x}_b$	Within	(= 0)	Within
Level-2 predictor: $L2x_b = \bar{x}_b - C_2$	(= 0)	Between	Between
Constant-centered level-1			
Level-1 predictor: $L1x_{wb} = x_{wb} - C_1$	Smushed	(= 0)	Within
Level-2 predictor: $L2x_{wb} = \bar{x}_b - C_2$	(= Within)	Between	Contextual

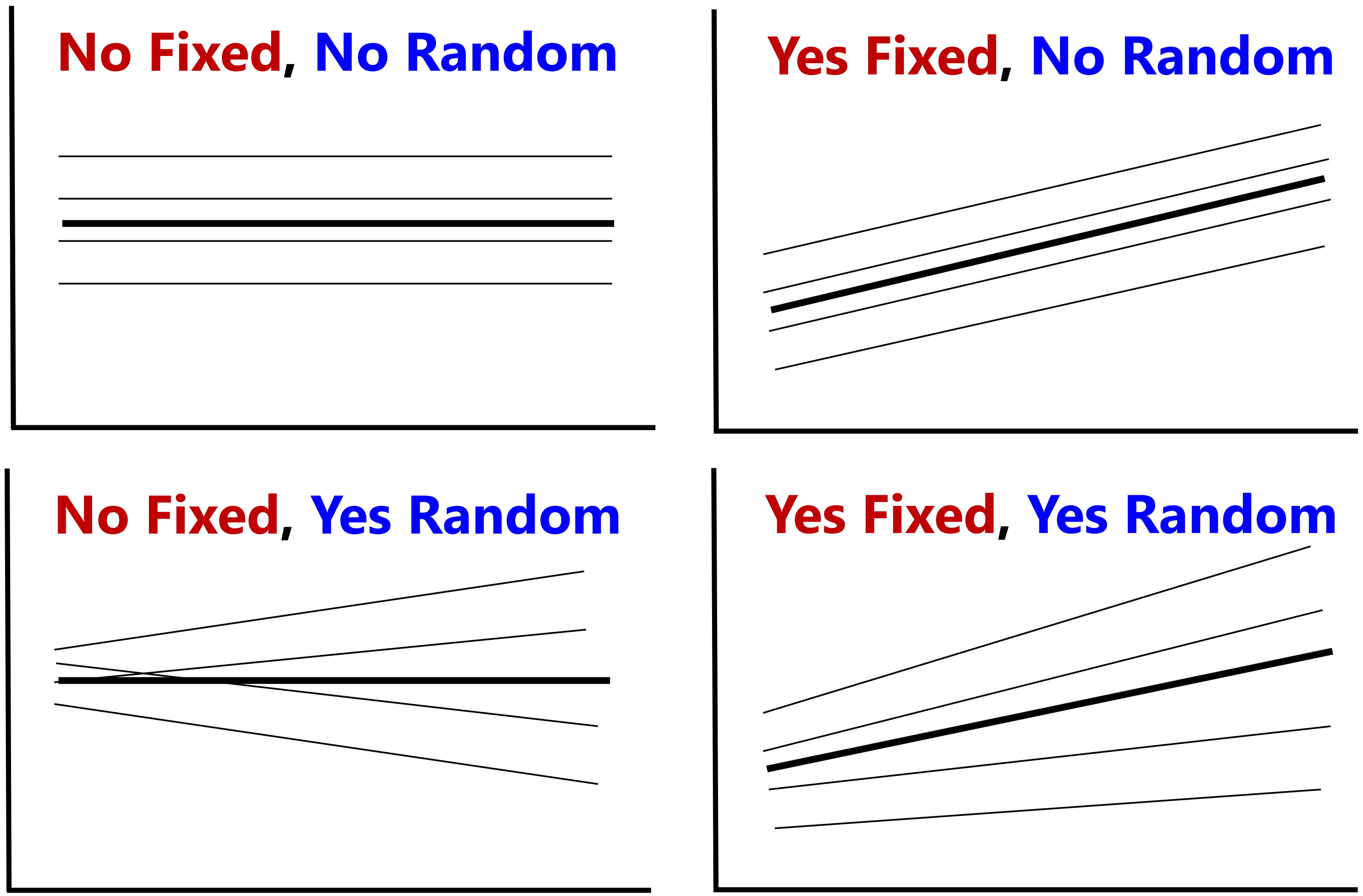
Abbreviations: w, within; b, between; C_1, level-1 centering constant; C_2, level-2 centering constant. Parentheses indicate assumptions about the fixed slopes of omitted predictors.

Constant-Centering for L1 predictors (+ Cluster Mean) may be preferable when:

- **You really do want level-2 contextual effects**
  - Directly model the incremental contribution of the cluster mean after controlling for a person's actual (not relative) predictor
- **For *categorical* level-1 predictors**
  - e.g., 0/1 predictors when cluster-MC → impossible values
- **When the cluster mean is not a reliable cluster-level predictor**
  - When the sample of persons within clusters is not complete enough to form a useful cluster mean, using externally-provided info may do a better job of representing the cluster (in which case cluster-MC doesn't really make sense without the cluster mean to go in with it)

- But cluster-MC or latent-centering is needed instead to prevent a L1 predictor's **random slope** from being smushed…
  - **Fixed slope** → every cluster gets the **same**
  - **Random slope** → every cluster gets their **own!**

Fixed and Random Effects of L1 Predictor
(Note: The cluster intercept is random in every figure)



Cluster-MC Predictor* with Random Slope

$L1x_{pc}$ is cluster-mean-centered into WCx_{pc}, with CMx_c at L2:

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(WCx_{pc}) + e_{pc}$

$WCx_{pc} = L1x_{pc} - \overline{L1x}_c \rightarrow$ only has L1 within variation

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10} + U_{1c}$

$CMx_c = \overline{L1x}_c - C_2 \rightarrow$ only has L2 between variation

U_{1c} is a random slope for the WC effect of WCx_{pc}

γ_{10} = within effect of having more $L1x_{pc}$ *than others in your cluster*

γ_{01} = between effect of having more $\overline{L1x}_c$ *than other clusters*

Because WCx_{pc} and CMx_c are uncorrelated, each gets the <u>total</u> effect for its level (L1 = within, L2 = between)

* If a constant-centered L1 predictor were used instead, the U_{1c} random slope would also multiply its L2 between part, creating bias in the estimated random slope variance. **To avoid such a smushed random slope, we need to use either cluster-MC (in univariate MLM) or latent-centering (in multivariate MLM).**

Adding Random L1 Cluster-MC Within Slope: (2b) Syntax by Univariate MLM Program

SAS:

```
PROC MIXED DATA=work.Example COVTEST NOCLPRINT IC METHOD=REML;
     CLASS schoolID;                           * GCORR = random effect correlations;
     MODEL langpost = hw2 mixgrd CMverb10 WCverb / GCORR SOLUTION DDFM=Satterthwaite;
     RANDOM INTERCEPT WCverb / TYPE=UN SUBJECT=schoolID;
     ESTIMATE "L2 Contextual Effect of Verbal"  CMverb10 1 WCverb -1;
RUN;
```

R lmer from lme4 package—using lmerTest package to get Satterthwaite denominator DF and contest1D:

```
name = lmer(data=Example, REML=TRUE,
            formula=langpost~1+hw2+mixgrd+CMverb10+WCverb+(1+WCverb|schoolID))
summary(name, ddf="Satterthwaite") # Shows random effect correlations already
contest1D(name, ddf="Satterthwaite", L=c(0,0,0,1,-1)) # L2 Contextual effect of verbal
```

STATA:

```
mixed langpost c.hw2 c.mixgrd c.CMverb10 c.WCverb, || schoolID: WCverb, ///
      covariance(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
estat recovariance, relevel(schoolID) correlation  // Random effect correlations
lincom c.CMverb10*1 + c.WCverb*-1, small  // L2 Contextual effect of verbal
```

SPSS:

```
MIXED langpost BY schoolID WITH hw2 mixgrd CMverb10 WCverb
      /METHOD   = REML
      /CRITERIA = DFMETHOD(SATTERTHWAITE)
      /PRINT    = SOLUTION TESTCOV
      /FIXED    = hw2 mixgrd CMverb10 WCverb
      /RANDOM   = INTERCEPT WCverb | COVTYPE(UN) SUBJECT(schoolID)
      /TEST     = "L2 Contextual effect of verbal" CMverb10 1 WCverb -1.
```

Example: Cluster-MC Random Slope

Level-1: $Lang_{pc} = \beta_{0c} + \beta_{1c}(Verbal_{pc} - \overline{Verbal}_c) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(HW_c - 2) + \gamma_{02}(MixGrd_c) + \gamma_{03}(\overline{Verbal}_c - 10) + U_{0c}$

$\beta_{1c} = \gamma_{10} + U_{1c}$

Adding L2 random slope variance of U_{1c} (as $\tau^2_{U_1}$) and L2 random intercept–slope covariance (as $\tau_{U_{01}}$)

Results from SAS MIXED:

L1 WCverb = $Verbal_{pc} - \overline{Verbal}_c$

L2 CMverb10 = $\overline{Verbal}_c - 10$

Solution for Fixed Effects					
Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	41.5281	0.3576	177	116.14	<.0001
hw2	-0.09509	0.4464	178	-0.21	0.8316
mixgrd	-0.9337	0.5052	201	-1.85	0.0660
CMverb10	3.6212	0.2647	209	13.68	<.0001
WCverb	2.4486	0.06831	151	35.85	<.0001

Btw, L2 Contextual = 1.173, SE = 0.273, p < .0001

Covariance Parameter Estimates					
Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	schoolID	8.4655	1.1352	[illegible]	[illegible]
UN(2,1)	schoolID	-0.6943	0.2386	[illegible]	[illegible]
UN(2,2)	schoolID	0.2239	0.08630	[illegible]	[illegible]
Residual		39.7586	0.9910	40.12	<.0001

Estimated G Correlation Matrix				
Row	Effect	schoolID	Col1	Col2
1	Intercept	1	1.0000	-0.5043
2	WCverb	1	-0.5043	1.0000

Likelihood ratio test of random slope variance (and intercept–slope covariance): $-2\Delta LL(\sim 2) = 19.29$, p < .0001

Implications of Random Slopes

- **L2 random slopes** capture a second, distinct source of cluster **dependency**—differences in **slope of a L1 person predictor**
  - Beyond **constant** covariance for persons from same L2 cluster (as created by the **L2 random intercept**), the **L2 random slope** adds **non-constant** covariance across values of its L1 predictor (e.g., $\boldsymbol{WCx_{pc}}$)
  - Also adds **quadratic heterogeneity** of variance across L1 predictor: $\boldsymbol{Var}(\boldsymbol{y_{pc}}) = \boldsymbol{\tau^2_{U_0}} + (\boldsymbol{WCx^2_{pc}} * \boldsymbol{\tau^2_{U_1}}) + (2\boldsymbol{WCx_{pc}} * \boldsymbol{\tau_{U_{01}}}) + \boldsymbol{\sigma^2_e}$

- **Random slopes do NOT*** explain variance (like **fixed slopes** do) because cluster slope differences are still "**error**" conceptually
  - We know **THAT** clusters need different slopes of **L1** $\boldsymbol{WCx_{pc}}$ but not **WHY**
- Therefore, random slopes imply another role for level-2 cluster predictors—to explain cluster differences in slope of **L1** $\boldsymbol{WCx_{pc}}$
  - To do so, we need "**cross-level interactions**" of L2 by L1 predictors!

** Hill that I will die on, but others disagree (i.e., marginal vs. conditional R^2)*

Adding Cross-Level Interactions: (2c) Syntax by Univariate MLM Program

SAS:

```
PROC MIXED DATA=work.Example COVTEST NOCLPRINT IC METHOD=REML;
     CLASS schoolID;                                  * In SAS, * creates interactions;
     MODEL langpost = hw2 mixgrd CMverb10 WCverb hw2*WCverb mixgrd*WCverb
                      CMverb10*WCverb / GCORR SOLUTION DDFM=Satterthwaite;
     RANDOM INTERCEPT WCverb / TYPE=UN SUBJECT=schoolID;
RUN;
```

R lmer from lme4 package—using lmerTest package to get Satterthwaite denominator DF:

```
name = lmer(data=Example, REML=TRUE,
            formula=langpost~1+hw2+mixgrd+CMverb10+WCverb+ hw2:WCverb +mixgrd:WCverb
                           +CMverb10:WCverb+(1+WCverb|schoolID))
summary(name, ddf="Satterthwaite") # In R, : creates interactions
```

STATA:

```
mixed langpost c.hw2 c.mixgrd c.CMverb10 c.WCverb c.hw2#c.WCverb c.mixgrd#c.WCverb ///
      c.CMverb10#c.WCverb, || schoolID: WCverb, /// In STATA, # creates interactions
      covariance(un) reml dfmethod(satterthwaite) dftable(pvalue) nolog
estat recovariance, relevel(schoolID) correlation  // Random effect correlations
```

SPSS: `* In SPSS, * creates interactions.`

```
MIXED langpost BY schoolID WITH hw2 mixgrd CMverb10 WCverb
      /METHOD    = REML
      /CRITERIA  = DFMETHOD(SATTERTHWAITE)
      /PRINT     = SOLUTION TESTCOV
      /FIXED     = hw2 mixgrd CMverb10 WCverb hw2*WCverb mixgrd*WCverb CMverb10*WCverb
      /RANDOM    = INTERCEPT WCverb | COVTYPE(UN) SUBJECT(schoolID).
```

Example: Adding Cross-Level Interactions

Level-1: $Lang_{pc} = \beta_{0c} + \beta_{1c}(Verbal_{pc} - \overline{Verbal_c}) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(HW_c - 2) + \gamma_{02}(MixGrd_c) + \gamma_{03}(\overline{Verbal_c} - 10) + U_{0c}$

$\beta_{1c} = \gamma_{10} + \gamma_{11}(HW_c - 2) + \gamma_{12}(MixGrd_c) + \gamma_{13}(\overline{Verbal_c} - 10) + U_{1c}$

Results from SAS MIXED—having more verbal ability than your peers matters more for your language score in schools with mixed grades:

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|
Intercept	41.5831	0.3629	172	114.58	<.0001
hw2	-0.04595	0.4590	179	-0.10	0.9204
mixgrd	-1.1368	0.5160	197	-2.20	0.0288
CMverb10	3.6445	0.2710	207	13.45	<.0001
WCverb	2.3903	0.1002	129	23.86	<.0001
hw2*WCverb	-0.05601	0.1305	143	-0.43	0.6683
mixgrd*WCverb	0.3210	0.1588	228	2.02	0.0444
CMverb10*WCverb	-0.04367	0.07805	182	-0.56	0.5765

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	schoolID	8.4680	1.1350	[illegible]	[illegible]
UN(2,1)	schoolID	-0.7095	0.2379	[illegible]	[illegible]
UN(2,2)	schoolID	0.2231	0.08640	[illegible]	[illegible]
Residual		39.7407	0.9903	40.13	<.0001

Relative to the previous model, the 3 cross-level interactions explained 0.04% of the L2 random WCverb slope variance

L1 WCverb slope is now specifically for hw=2, mixgrd=0, and CMverb=10; those 3 slopes are now specifically for WCverb=0 (school mean)

L1 WCverb slope is significantly more positive (stronger) in schools with mixed grades (and nonsignificantly weaker in schools with more homework and higher mean verbal ability).

Explained Variance by Fixed Slopes

- **Fixed slopes of level-2 cluster predictors *by themselves*:**
  - L2 BC main effects or interactions reduce L2 random intercept variance
- **Fixed slopes of *cross-level interactions* (level-1 * level-2):**
  - If the **L1 person predictor also has a random slope**, its cross-level interaction will reduce its corresponding **L2 random slope variance**
    - So make sure you test the random slope before any cross-level interactions!
  - If the **L1 person predictor does NOT have a random slope**, its cross-level interaction will reduce the **L1 residual variance** instead
    - This condition creates a "systematically varying" L1 slope instead, in which the slope varies only by interacting predictors (but not randomly otherwise)
- **Fixed slopes of level-1 person predictors *without L2 variance*:**
  - L1 WC main effects or interactions reduce L1 residual variance
- **Fixed slopes of level-1 person predictors *with L2 variance*:**
  - L1 WC main effects or interactions reduce both L1 residual variance and L2 random intercept variance; need to add corresponding L2 main effects, L2 interactions, or cross-level interactions in order to prevent smushing!
    - See Hoffman & Walters (2022) and Hoffman (2019) for elaboration

Part 2: Summary

- **Level-1 predictors** are person characteristics, but they **almost always contain cluster mean differences** (level-2 variance) as well
  - **Variance** at each level → **different slope** at each level!

- **3 options** for specifying fixed slopes of a L1 predictor in order to distinguish its level-specific effects (i.e., **avoid smushed effects**):
  1. **Cluster-Mean-Centering**: Manually carve up into L2 BC (cluster mean → **L2 Between slope**) and L1 WC deviation (→ **L1 Within slope**)
  2. **Constant-Centering**: Add cluster mean to become **L2 Contextual slope**, then L1 predictor's unique effect is **L1 Within slope**
  3. **Latent-Centering**: Let multivariate MLM estimate L2 and L1 variance components, same as for the outcome → analogous to Cluster-MC

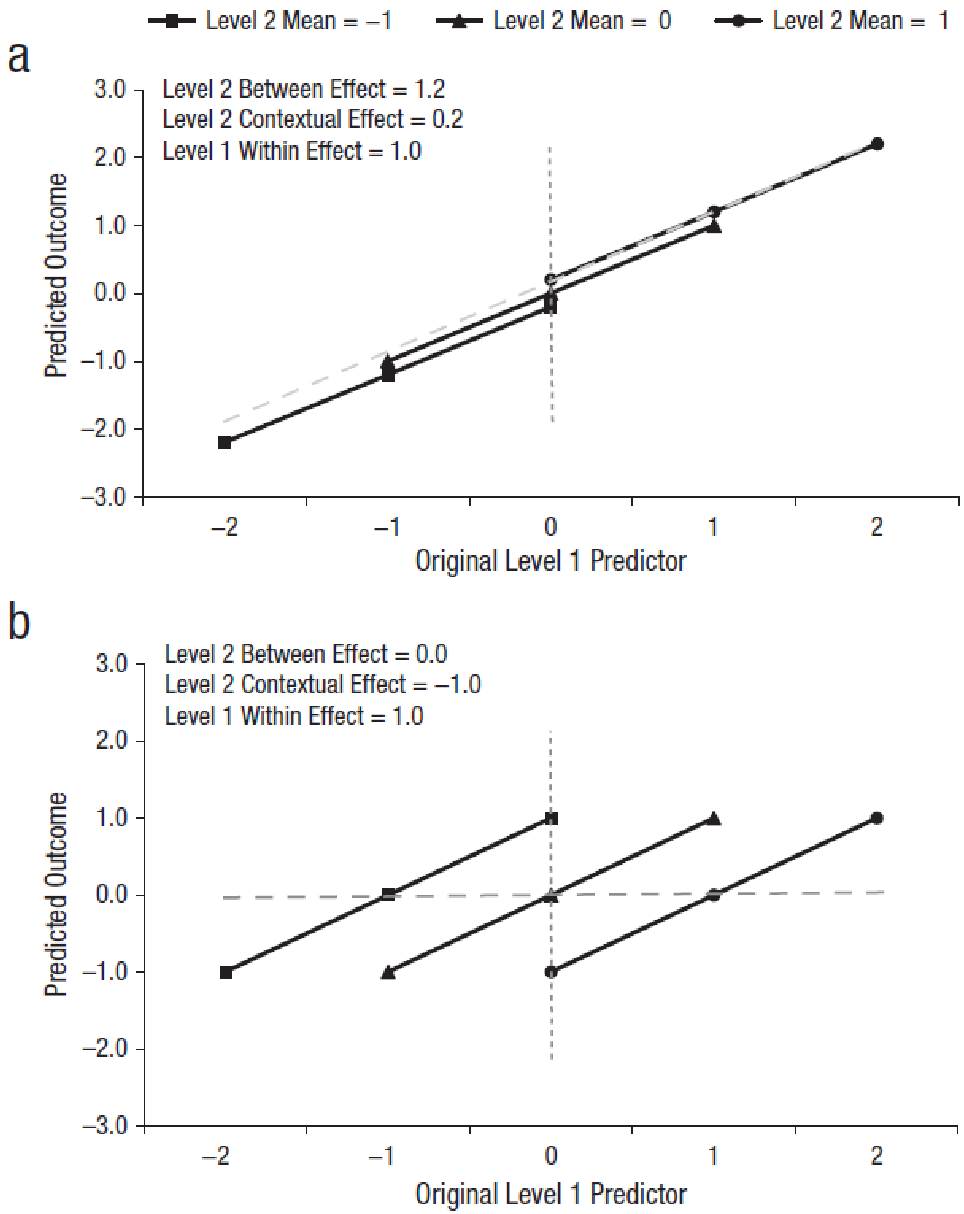
- A **level-2 random slope** variance allows cluster differences in the effect of a L1 person predictor (using only options 1 or 3)
  - Implies **heterogeneity** of variance and covariance across L1 predictor
  - Implies **another way clusters differ** (to be explained by **cross-level interactions** between that L1 predictor and L2 predictors)

Bonus Material

- Comparison of between vs contextual effects

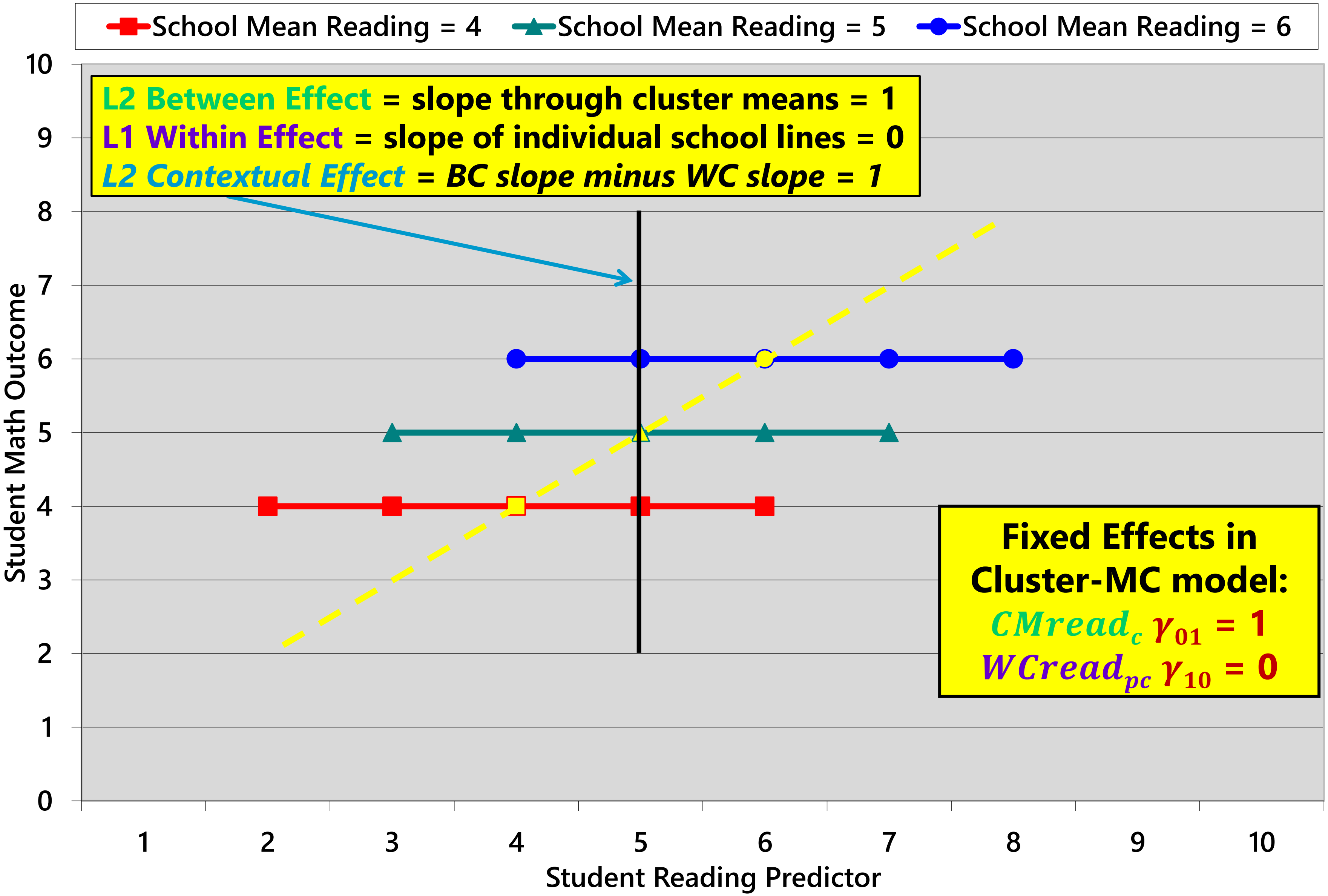
- More depictions of level-2 between, level-2 contextual, and level-1 within slopes
  - Example variables: How often students are read to by their parents predict student math outcomes

Bonus: Between vs. Contextual Effects

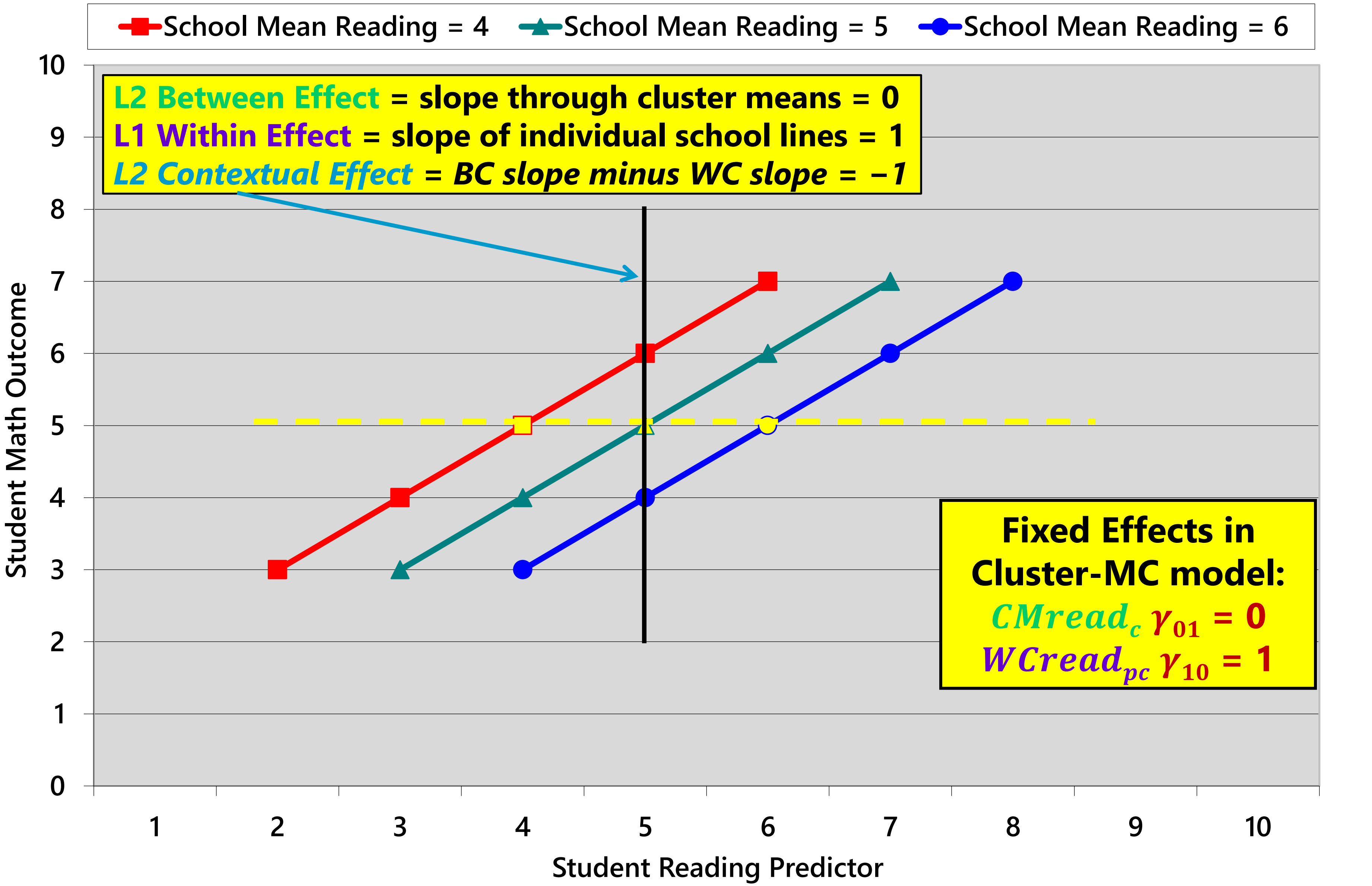


- Image from Hoffman (2019), example using student SES
- *Top:* Contextual effect is minimal—there is no added benefit to going to a high-SES school when comparing across schools *at same level of student SES*
- *Bottom:* Contextual effect is negative—at the same student SES level, relatively high students from low-SES schools do better than relatively low students from high-SES schools

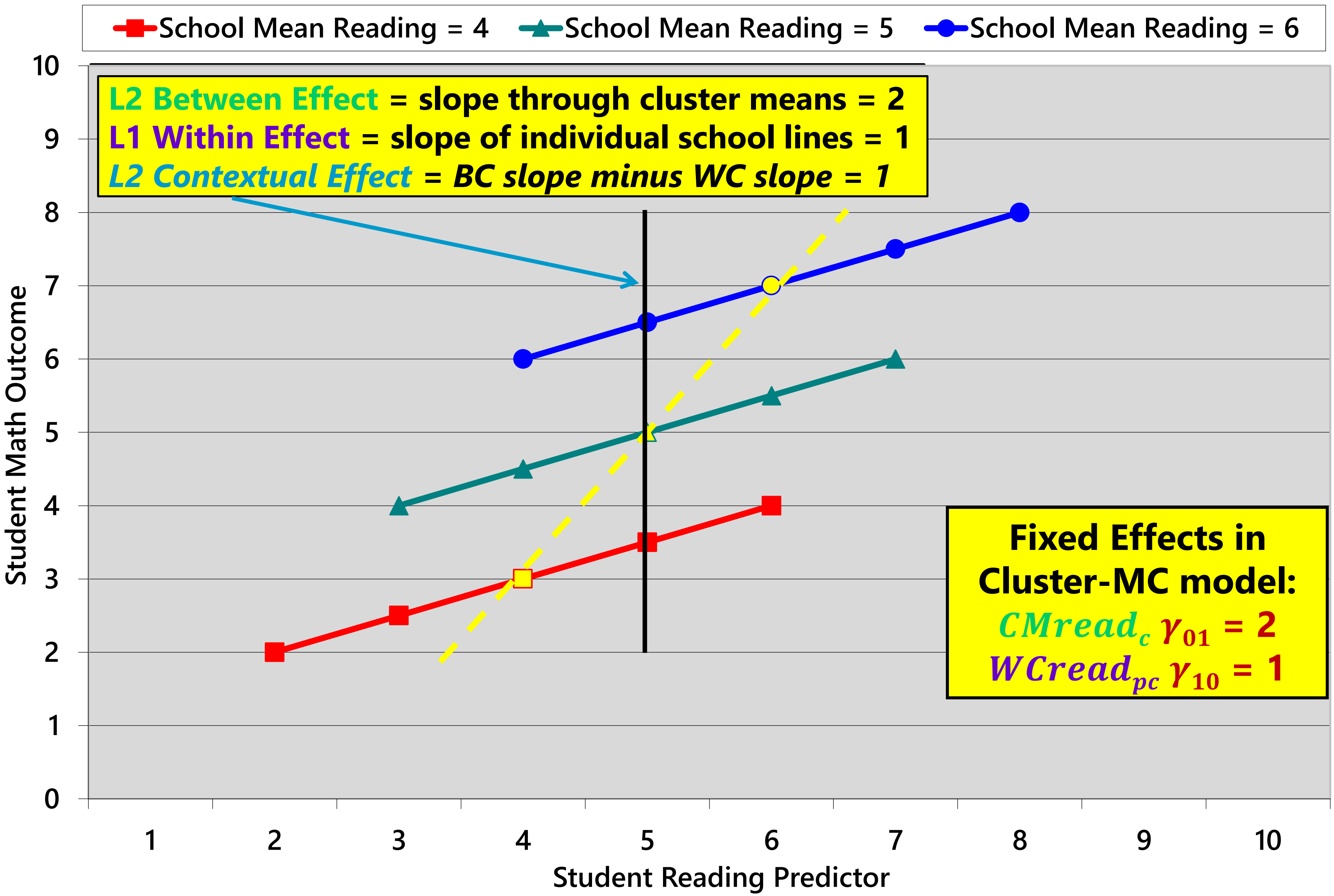
ALL Between Effect, **NO** Within Effect



NO Between Effect, **ALL** Within Effect



Between Effect > Within Effect



Between, Within, and Contextual Effects