Time-Varying Predictors for Within-Person Fluctuation

- Today's topics:
  - Review of time-invariant predictors
  - Time-varying predictors that fluctuate over time
  - Person-Mean-Centering (PMC)
  - Grand-Mean-Centering (GMC)
  - Model extensions under Person-MC vs. Grand-MC
  - Model assumptions
  - Predicting heterogeneity of variance

Modeling Time-Invariant Predictors

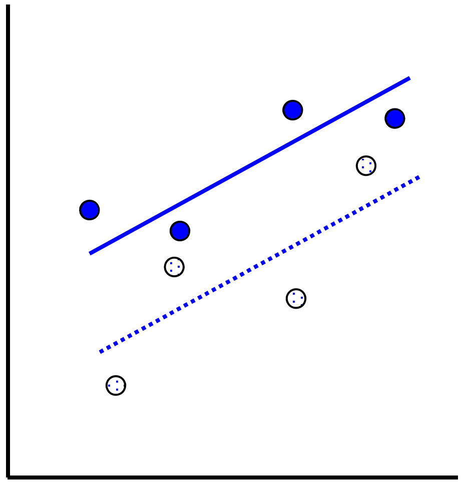
What independent variables can be time-invariant predictors?

- Also known as "person-level" or "level-2" predictors
- Include substantive predictors, controls, and predictors of missingness
- Can be anything that **does not change across time** (e.g., Biological Sex)
- Can be anything that **is not likely to change across the study,** but you may have to argue for this (e.g., Parenting Strategies, SES)
- Can be anything that **does change across the study**…
  - But you have **only measured once**
    - Limit conclusions to variable's status at time of measurement
    - e.g., "Parenting Strategies at age 10"
  - Or **is perfectly correlated with time** (age, time to event)
    - Would use Age at Baseline, or Time to Event *from Baseline* instead

The Role of Time-Invariant Predictors in the **Model for the Means**

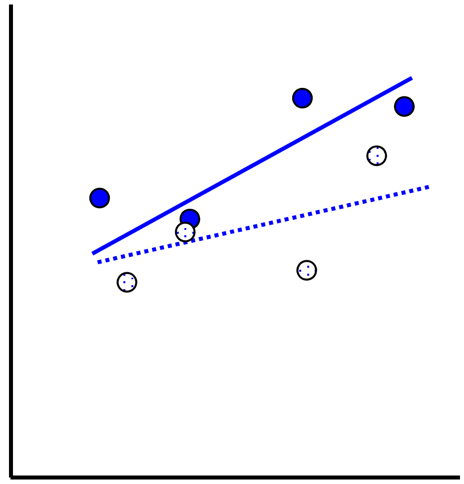
- **In Within-Person Change Models** → Adjust growth curve

Main effect of X, No interaction with time



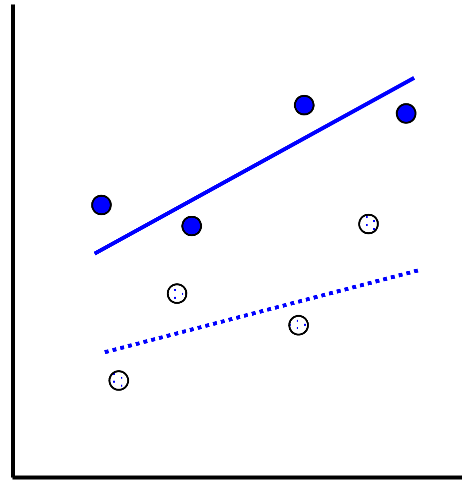
← Time →

Interaction with time, Main effect of X?



← Time →

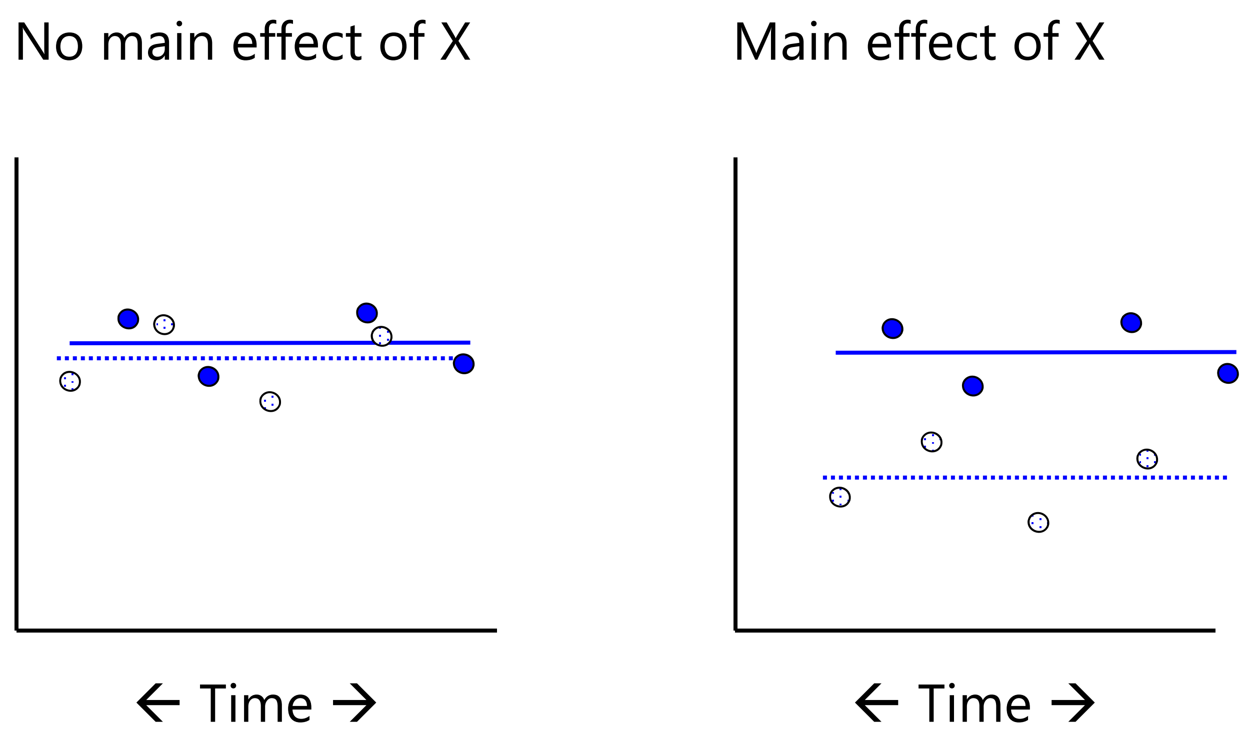
Main effect of X, and Interaction with time



← Time →

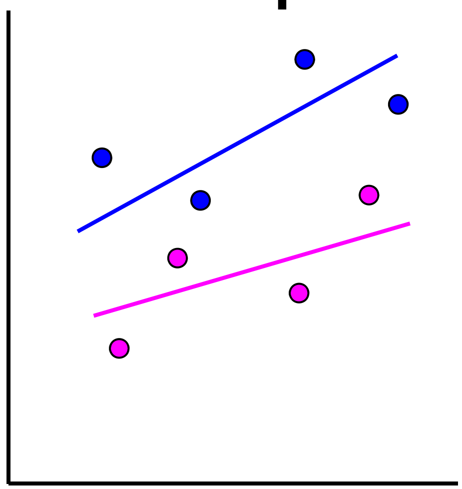
The Role of Time-Invariant Predictors in the **Model for the Means**

- In **Within-Person Fluctuation Models** → Adjust mean level



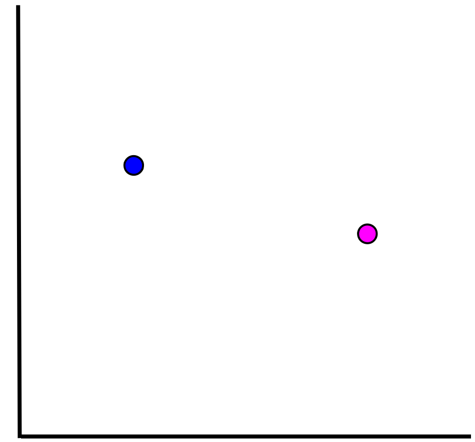
Why Level-2 Predictors Cannot Have Random Effects in 2-Level Models

Random Slopes for Time



Time
(or Any Level-1 Predictor)

Random Slopes for Sex?



Sex
(or any Level-2 Predictor)

You cannot make a line out of a dot, so level-2 effects cannot vary randomly over persons.

Education as a Time-Invariant Predictor:
Example using a Random Quadratic Time Model

- <u>Main Effect of Education = Education*Intercept Interaction</u>
  - Moderates the intercept → Increase or decrease in expected outcome at time 0 for every year of education

- <u>Effect of Education on Linear Time = Education*Time Interaction</u>
  - Moderates the linear time slope → Increase or decrease in expected rate of change at time 0 for every year of education

- <u>Effect of Education on Quadratic Time = Education*$Time^2$ Interaction</u>
  - Moderates the quadratic time slope → Increase or decrease in half of expected acceleration/deceleration of linear rate of change for every year of education

Education (12 years = 0) as a Time-Invariant Predictor: Example using a Random Quadratic Time Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + \gamma_{01}Ed_i + U_{0i}$

- β_{0i}: **Intercept** for person *i*
- γ_{00}: **Fixed Intercept when Time=0 and Ed=12**
- γ_{01}: **Δ in Intercept per unit Δ in Ed**
- U_{0i}: **Random (Deviation) Intercept after controlling for Ed**

$\beta_{1i} = \gamma_{10} + \gamma_{11}Ed_i + U_{1i}$

- β_{1i}: **Linear Slope** for person *i*
- γ_{10}: **Fixed Linear Time Slope when Time=0 and Ed=12**
- γ_{11}: **Δ in Linear Time Slope per unit Δ in Ed (=Ed*time)**
- U_{1i}: **Random (Deviation) Linear Time Slope after controlling for Ed**

$\beta_{2i} = \gamma_{20} + \gamma_{21}Ed_i + U_{2i}$

- β_{2i}: **Quad Slope** for person *i*
- γ_{20}: **Fixed Quad Time Slope when Ed = 12**
- γ_{21}: **Δ in Quad Time Slope per unit Δ in Ed (=Ed*time²)**
- U_{2i}: **Random (Deviation) Quad Time Slope after controlling for Ed**

Education (12 years = 0) as a Time-Invariant Predictor: Example using a Random Quadratic Time Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}$

Level 2 Equations (one per β):

$$\beta_{0i} = \gamma_{00} + \gamma_{01}Ed_i + U_{0i}$$

$$\beta_{1i} = \gamma_{10} + \gamma_{11}Ed_i + U_{1i}$$

$$\beta_{2i} = \gamma_{20} + \gamma_{21}Ed_i + U_{2i}$$

γ_{11} and γ_{21} are known as "**cross-level**" interactions (level-1 predictor by level-2 predictor)

- Composite equation:
- $y_{ti} = (\gamma_{00} + \gamma_{01}Ed_i + U_{0i}) + (\gamma_{10} + \gamma_{11}Ed_i + U_{1i})Time_{ti} + (\gamma_{20} + \gamma_{21}Ed_i + U_{2i})Time_{ti}^2 + e_{ti}$

Fixed Effects of Time-Invariant Predictors

- Question of interest: Why do people change differently?
  - We're trying to predict individual differences in intercepts and slopes (i.e., reduce level-2 random effects variances)
  - So level-2 random effects variances become 'conditional' on predictors → actually random effects variances *left over*

$$\begin{aligned} \beta_{0i} &= \gamma_{00} + U_{0i} \\ \beta_{1i} &= \gamma_{10} + U_{1i} \\ \beta_{2i} &= \gamma_{20} + U_{2i} \end{aligned} \quad \longrightarrow \quad \begin{aligned} \beta_{0i} &= \gamma_{00} + \gamma_{01}Ed_i + U_{0i} \\ \beta_{1i} &= \gamma_{10} + \gamma_{11}Ed_i + U_{1i} \\ \beta_{2i} &= \gamma_{20} + \gamma_{21}Ed_i + U_{2i} \end{aligned}$$

  - Can calculate pseudo-R^2 for each level-2 random effect variance between models with *fewer* versus *more* parameters as:

$$\text{Pseudo } R^2 = \frac{\text{random variance}_{\text{fewer}} - \text{random variance}_{\text{more}}}{\text{random variance}_{\text{fewer}}}$$

Fixed Effects of Time-Invariant Predictors

- What about predicting level-1 effects with no random variance?
  - If the random linear time slope is n.s., can I test interactions with time?

This should be ok to do…

$$\beta_{0i} = \gamma_{00} + \gamma_{01}Ed_i + U_{0i}$$
$$\beta_{1i} = \gamma_{10} + \gamma_{11}Ed_i + U_{1i}$$
$$\beta_{2i} = \gamma_{20} + \gamma_{21}Ed_i + U_{2i}$$

Is this still ok to do?

$$\beta_{0i} = \gamma_{00} + \gamma_{01}Ed_i + U_{0i}$$
$$\beta_{1i} = \gamma_{10} + \gamma_{11}Ed_i$$
$$\beta_{2i} = \gamma_{20} + \gamma_{21}Ed_i$$

  - YES, surprisingly enough….
  - **In theory**, if a level-1 effect does not vary randomly over individuals, then it has "no" variance to predict (so cross-level interactions with that level-1 effect are not necessary)
  - However, because power to detect random effects is often lower than power to detect fixed effects, fixed effects of predictors can still be significant even if there is "no" (≈0) variance for them to predict
  - Just make sure you test for random effects BEFORE testing any cross-level interactions with that level-1 predictor!

3 Types of Effects: Fixed, Random, and Systematically (Non-Randomly) Varying

Let's say we have a significant fixed linear effect of time. What happens after we test a sex*time interaction?

	Non-Significant Sex*Time effect?	Significant Sex*Time effect?
Random time slope initially **not** significant	Linear effect of time is **FIXED**	Linear effect of time is **systematically varying**
Random time initially sig, **not** sig. after sex*time	---	Linear effect of time is **systematically varying**
Random time initially sig, **still** sig. after sex*time	Linear effect of time is **RANDOM**	Linear effect of time is **RANDOM**

The effects of level-1 predictors (time-level) can be fixed, random, or systematically varying. The effects of level-2 predictors (person-level) can only be fixed or systematically varying (nothing to be random over…yet).

Variance Accounted For By Level-2 Time-Invariant Predictors

- **Fixed effects of level 2 predictors *by themselves*:**
  - L2 (BP) main effects (e.g., sex) reduce L2 (BP) random intercept variance
  - L2 (BP) interactions (e.g., sex by ed) also reduce L2 (BP) random intercept variance

- **Fixed effects of *cross-level interactions* (level 1* level 2):**
  - If the interacting level-1 predictor is random, any cross-level interaction with it will reduce its corresponding level-2 BP random slope variance
    - e.g., if *time* is random, then sex**time*, ed**time*, and sex*ed**time* can each reduce the random linear time slope variance
  - If the interacting level-1 predictor not random, any cross-level interaction with it will reduce the level-1 WP residual variance instead
    - e.g., if $time^2$ is fixed, then sex*$time^2$, ed*$time^2$, and sex*ed*$time^2$ will reduce the L1 (WP) residual variance → Different quadratic slopes from sex and ed will allow better trajectories, reduce the variance around trajectories

Variance Accounted for… For Real

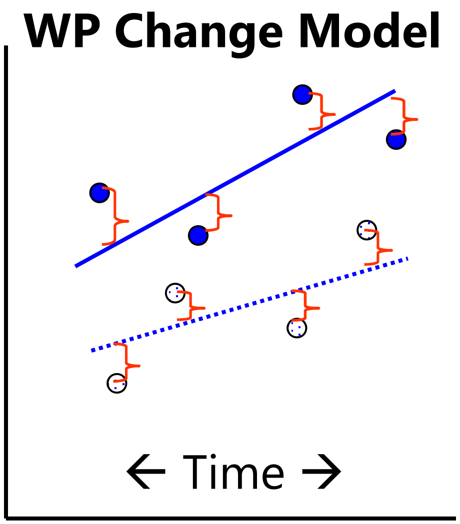
- **Pseudo-R^2** is named that way for a reason… piles of variance can shift around, such that it can actually be negative
  - Sometimes a sign of model mis-specification
  - Hard to explain to readers when it happens!

- **One last simple alternative: Total R^2**
  - Generate model-predicted y's from fixed effects only (NOT including random effects) and correlate with observed y's
  - Then square correlation → total R^2
  - Total R^2 = total reduction in overall variance of y across levels
  - Can be "unfair" in models with large unexplained sources of variance

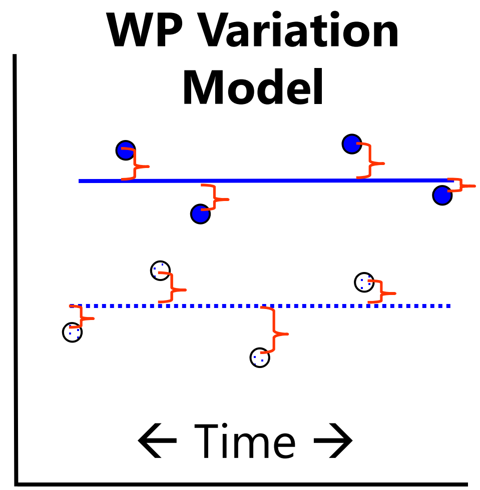
- MORAL OF THE STORY: Specify EXACTLY which kind of pseudo-R^2 you used—give the formula and the reference!!

The Joy of Time-Varying Predictors

- TV predictors predict leftover **WP (residual) variation**:



If model for time works, then residuals should look like this →



- Modeling time-varying predictors is complicated because they represent an **aggregated effect**:
  - Effect of the *between-person* variation in the predictor x_{ti} on Y
  - Effect of the *within-person* variation in the predictor x_{ti} on Y
  - Here we are assuming the predictor x_{ti} only **fluctuates** over time…
    - *We will need a different model if x_{ti} changes systematically over time…*

The Joy of Time-Varying Predictors

- Time-varying (TV) predictors usually carry 2 kinds of effects because they are really 2 predictor variables, not 1
- Example: Stress measured daily
  - Some days are worse than others:
    - **WP variation in stress** *(represented as deviation from own mean)*
  - Some people just have more stress than others all the time:
    - **BP variation in stress** *(represented as person mean predictor over time)*
- Can quantify each source of variation with an ICC
  - ICC = (BP variance) / (BP variance + WP variance)
  - ICC > 0? TV predictor has BP variation (so it *could* have a BP effect)
  - ICC < 1? TV predictor has WP variation (so it *could* have a WP effect)

Between-Person vs. Within-Person Effects

- Between-person and within-person effects in <u>SAME</u> direction
  - Stress → Health?
    - **BP: People with more chronic stress than other people may have <u>worse</u> general health than people with less chronic stress**
    - **WP: People may feel <u>worse</u> than usual when they are currently under more stress than usual (regardless of what “usual” is)**
- Between-person and within-person effects in <u>OPPOSITE</u> directions
  - Exercise → Blood pressure?
    - **BP: People who exercise more often generally have <u>lower</u> blood pressure than people who are more sedentary**
    - **WP: During exercise, blood pressure is <u>higher</u> than during rest**
- Variables have different **meanings** at different levels!
- Variables have different **scales** at different levels

3 Kinds of Effects for TV Predictors

- **Is the Between-Person (BP) effect significant?**
  - Are people with higher predictor values than other people (*on average over time*) also higher on Y than other people (*on average over time*), such that the person mean of the TV predictor accounts for level-2 random intercept variance ($\tau^2_{U_0}$)?

- **Is the Within-Person (WP) effect significant?**
  - If you have higher predictor values than usual (*at this occasion*), do you also have higher outcomes values than usual (*at this occasion*), such that the within-person deviation of the TV predictor accounts for level-1 residual variance (σ^2_e)?

- **Are the BP and WP effects different sizes: Is there a contextual effect?**
  - After controlling for the absolute value of TV predictor at each occasion, is there still an incremental contribution from having a higher person mean of the TV predictor (i.e., does one's general tendency predict $\tau^2_{U_0}$ above and beyond)?
  - If there is no contextual effect, then the BP and WP effects of the TV predictor show *convergence*, such that their effects are of equivalent magnitude

Modeling TV Predictors (labeled as x_{ti})

- **Level-2 effect of x_{ti}:**
  - The level-2 effect of x_{ti} is usually represented by the person's mean of time-varying x_{ti} across time (labeled as **PMx$_i$** or $\bar{\mathbf{X}}_\mathbf{i}$)
  - PMx$_i$ should be centered at a CONSTANT (grand mean or other) so that 0 is meaningful, just like any other time-invariant predictor

- **Level-1 effect of x_{ti} can be included two different ways:**
  - "**Group-mean-centering**" → "**person-mean-centering**" in longitudinal, in which level-1 predictors are centered using a level-2 VARIABLE
  - "**Grand-mean-centering**" → level-1 predictors are centered using a CONSTANT (not necessarily the grand mean; it's just called that)
  - Note that these 2 choices do NOT apply to the level-2 effect of x_{ti}!
    - But the interpretation of the level-2 effect of x_{ti} WILL DIFFER based on which centering method you choose for the level-1 effect of x_{ti}!

Person-Mean-Centering (P-MC)

- In P-MC, we decompose the TV predictor x_{ti} into 2 variables that directly represent its BP (level-2) and WP (level-1) sources of variation, and include those variables as the predictors instead:

- **Level-2, PM predictor = person mean of $\mathbf{x_{ti}}$**
  - $\mathbf{PMx_i = \bar{X}_i - \boldsymbol{C}}$
  - $\mathrm{PMx_i}$ is centered at a constant C, chosen so 0 is meaningful
  - $\mathrm{PMx_i}$ is positive? Above sample mean → "more than other people"
  - $\mathrm{PMx_i}$ is negative? Below sample mean → "less than other people"

- **Level-1, WP predictor = deviation from person mean of $\mathbf{x_{ti}}$**
  - $\mathbf{WPx_{ti} = x_{ti} - \bar{X}_i}$ *(note: uncentered person mean $\boldsymbol{\bar{X}_i}$ is used to center x_{ti})*
  - $\mathrm{WPx_{ti}}$ is NOT centered at a constant; is centered at a VARIABLE
  - $\mathrm{WPx_{ti}}$ is positive? Above your own mean → "more than usual"
  - $\mathrm{WPx_{ti}}$ is negative? Below your own mean → "less than usual"

Within-Person Fluctuation Model with **Person-Mean-Centered Level-1 x_{ti}**

→ WP and BP Effects directly through **separate** parameters

x_{ti} is person-mean-centered into WPx_{ti}, with PMx_i at L2:

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(\mathbf{WPx_{ti}}) + \mathbf{e_{ti}}$

$WPx_{ti} = x_{ti} - \bar{X}_i$ → it has only Level-1 WP variation

Level 2: $\beta_{0i} = \boldsymbol{\gamma_{00}} + \boldsymbol{\gamma_{01}}(\mathbf{PMx_i}) + \mathbf{U_{0i}}$

$\beta_{1i} = \boldsymbol{\gamma_{10}}$

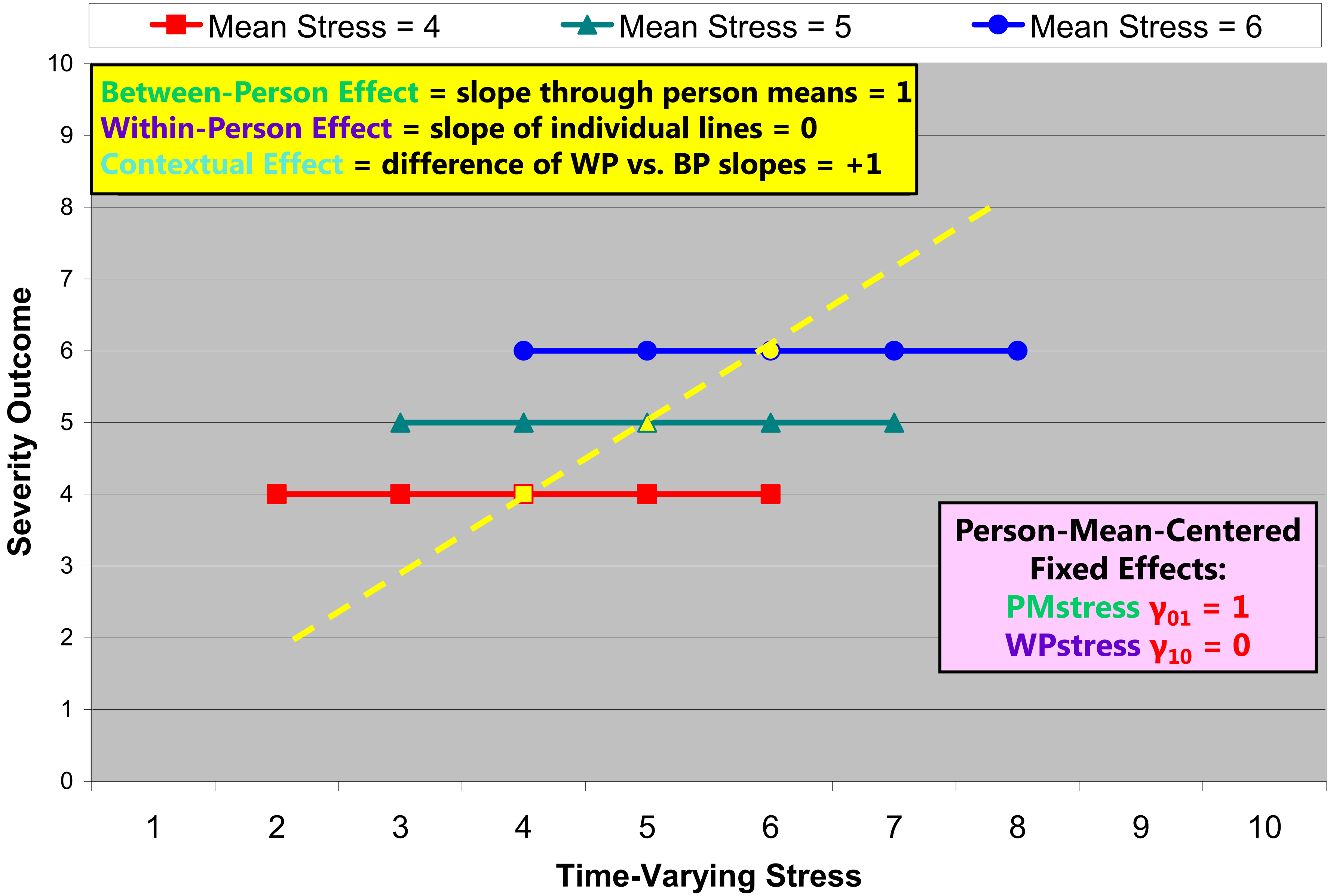
$PMx_i = \bar{X}_i - C$ → it has only Level-2 BP variation

γ_{10} = WP main effect of having more x_{ti} than usual

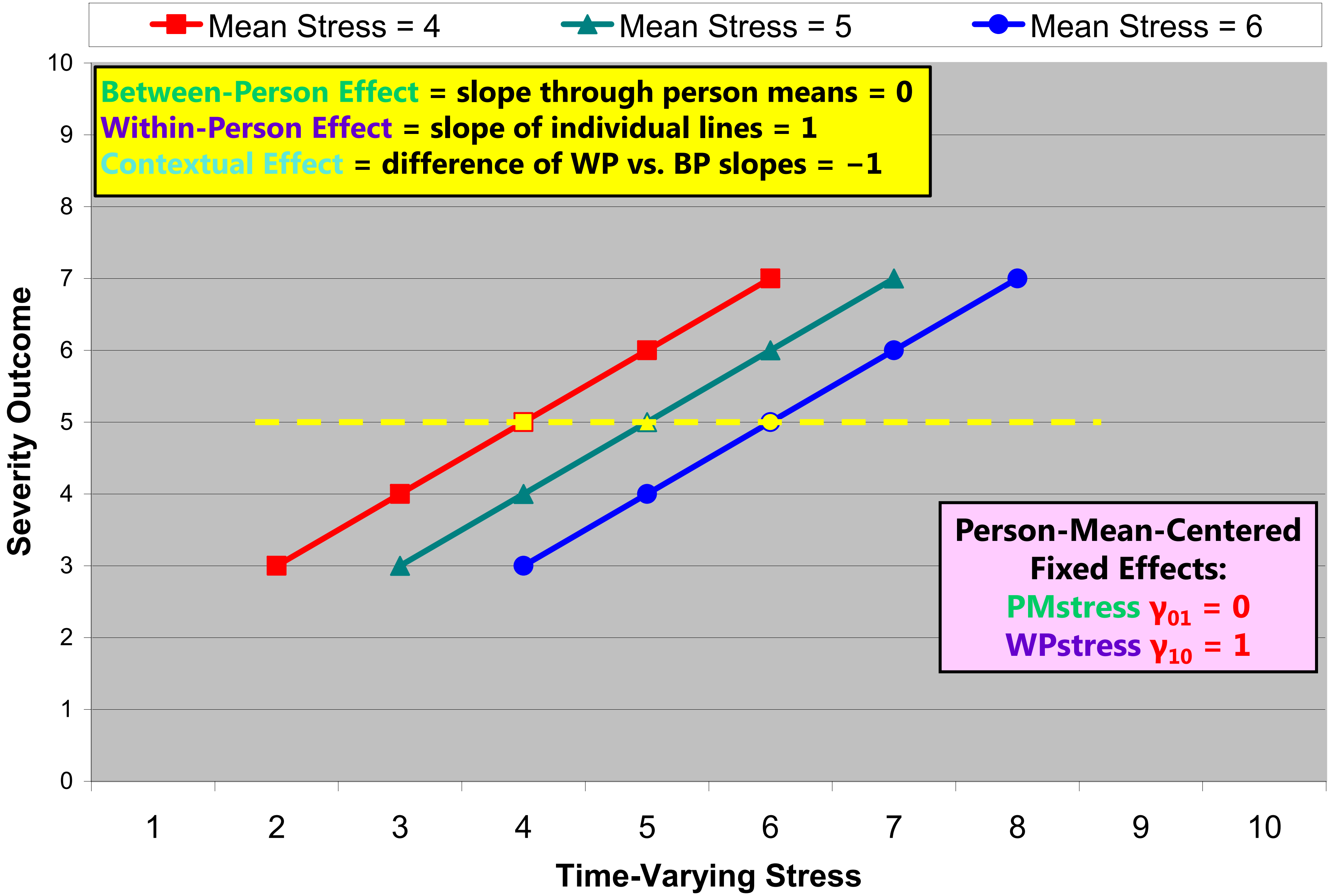
γ_{01} = BP main effect of having more $\bar{X}_i$ than other people

Because WPx_{ti} and PMx_i are uncorrelated, each gets the total effect for its level (WP=L1, BP=L2)

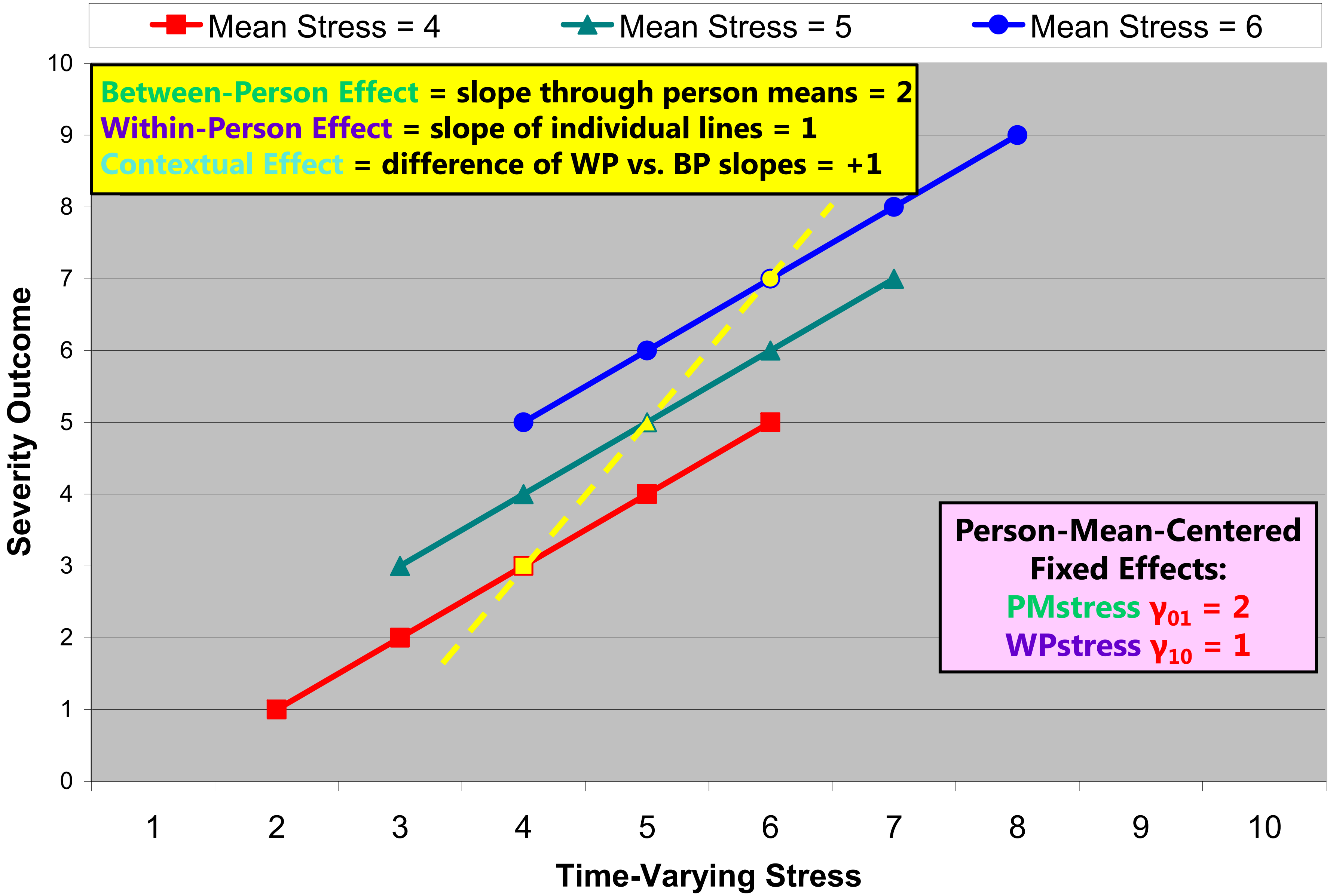
ALL Between-Person Effect, NO Within-Person Effect



NO Between-Person Effect, ALL Within-Person Effect



Between-Person Effect > Within-Person Effect



Within-Person Fluctuation Model with **Person-Mean-Centered Level-1 x_{ti}**

→ WP and BP Effects directly through **separate** parameters

x_{ti} is person-mean-centered into WPx_{ti}, with PMx_i at L2:

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(WPx_{ti}) + e_{ti}$

$WPx_{ti} = x_{ti} - \bar{X}_i$ → it has only Level-1 WP variation

$PMx_i = \bar{X}_i - C$ → it has only Level-2 BP variation

Level 2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + \gamma_{11}(PMx_i) + U_{1i}$

U_{1i} is a random slope for the WP effect of x_{ti}

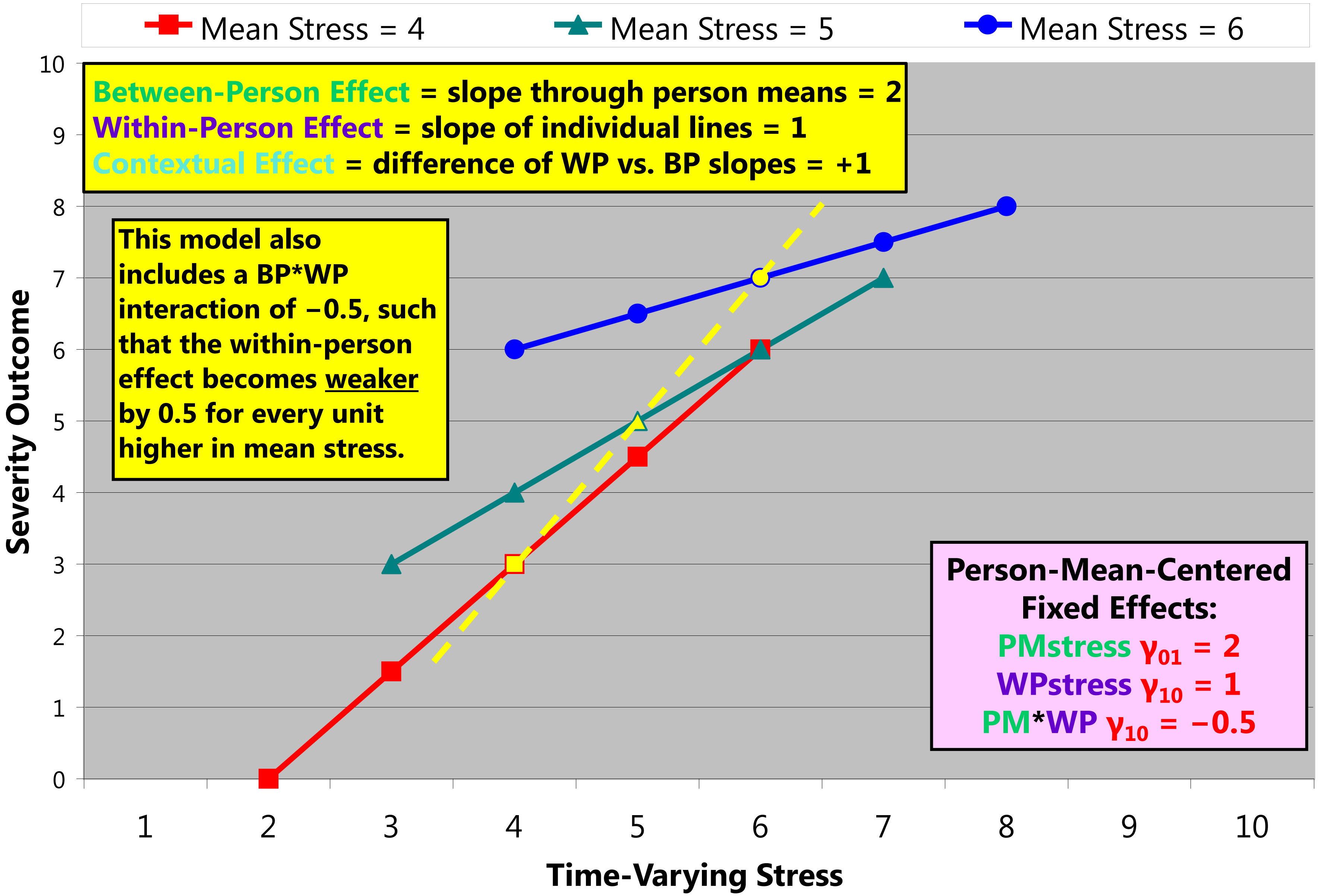
γ_{10} = WP simple main effect of having more x_{ti} than usual *for* $PMx_i = 0$

γ_{01} = BP simple main effect of having more $\bar{X}_i$ than other people for people at their own mean ($WPx_{ti} = x_{ti} - \bar{X}_i$ → 0)

γ_{11} = BP*WP interaction: how the effect of having more x_{ti} than usual differs by how much $\bar{X}_i$ you have

Note: this model should also test γ_{02} for $PMx_i * PMxi$ (stay tuned)

Between-Person x Within-Person Interaction



3 Kinds of Effects for TV Predictors

- **What Person-Mean-Centering tells us <u>directly</u>:**

- **Is the Between-Person (BP) effect significant?**
  - Are people with higher predictor values <u>than other people</u> (*on average over time*) also higher on Y <u>than other people</u> (*on average over time*), such that the person mean of the TV predictor accounts for level-2 random intercept variance ($\tau^2_{U_0}$)?
  - This would be indicated by a significant fixed effect of $\mathbf{PMx_i}$
  - Note: this is NOT controlling for the absolute value of x_{ti} at each occasion

- **Is the Within-Person (WP) effect significant?**
  - If you have higher predictor values <u>than usual</u> (*at this occasion*), do you also have higher outcomes values <u>than usual</u> (*at this occasion*), such that the within-person deviation of the TV predictor accounts for level-1 residual variance (σ^2_e)?
  - This would be indicated by a significant fixed effect of $\mathbf{WPx_{ti}}$
  - Note: this is represented by the <u>relative</u> value of x_{ti}, NOT the <u>absolute</u> value of x_{ti}

3 Kinds of Effects for TV Predictors

- **What Person-Mean-Centering DOES NOT tell us <u>directly</u>:**

- **Are the BP and WP effects different sizes: Is there a contextual effect?**
  - After controlling for the absolute value of the TV predictor at each occasion, is there still <u>an incremental contribution from having a higher person mean of the TV predictor</u> (i.e., does one's general tendency predict $\tau^2_{U_0}$ above and beyond just the time-specific value of the predictor)?
  - If there is no contextual effect, then the BP and WP effects of the TV predictor show ***<u>convergence</u>***, such that their effects are of equivalent magnitude

- **To answer this question about the contextual effect for the incremental contribution of the person mean, we have two options:**
  - Ask for the contextual effect via an ESTIMATE statement in SAS (or TEST in SPSS, or NEW in Mplus, or LINCOM in STATA): $\mathbf{WPx_{ti}}$ -1 $\mathbf{PMx_{i}}$ 1
  - Use **"<u>grand-mean-centering</u>"** for time-varying x_{ti} instead: $\mathbf{TVx_{ti} = x_{ti} - \boldsymbol{C}}$
    → centered at a CONSTANT, NOT A LEVEL-2 VARIABLE
    - Which constant only matters for what the reference point is; it could be the grand mean or other

Remember Regular Old Regression?

- In this model: $Y_i = \beta_0 + \beta_1(X_{1i}) + \beta_2(X_{2i}) + e_i$

  - If X_{1i} and X_{2i} **ARE NOT** correlated:
    - β_1 is **ALL the relationship** between X_{1i} and Y_i
    - β_2 is **ALL the relationship** between X_{2i} and Y_i

  - If X_{1i} and X_{2i} **ARE** correlated:
    - β_1 is **different than** the full relationship between X_{1i} and Y_i
      - "Unique" effect of X_{1i} *controlling for* X_{2i} or *holding* X_{2i} *constant*
    - β_2 is **different than** the full relationship between X_{2i} and Y_i
      - "Unique" effect of X_{2i} *controlling for* X_{1i} or *holding* X_{1i} *constant*

  - Hang onto that idea…

Person-MC vs. Grand-MC for Time-Varying Predictors

Level 2		Original	Person-MC Level 1	Grand-MC Level 1
$\bar{X}_i$	$PMx_i = \bar{X}_i - 5$	x_{ti}	$WPx_{ti} = x_{ti} - \bar{X}_i$	$TVx_{ti} = x_{ti} - 5$
3	−2	2	−1	−3
3	−2	4	1	−1
7	2	6	−1	1
7	2	8	1	3

Same PMx_i goes into the model using either way of centering the level-1 variable x_{ti}

Using **Person-MC**, WPx_{ti} has NO level-2 BP variation, so it is not correlated with PMx_i

Using **Grand-MC**, TVx_{ti} STILL has level-2 BP variation, so it is STILL CORRELATED with PMx_i

So the effects of PMx_i and TVx_{ti} when included together under Grand-MC will be different than their effects would be if they were by themselves…

Within-Person Fluctuation Model with $\mathbf{x_{ti}}$ represented at Level 1 Only:
→ WP and BP Effects are **Smushed Together**

x_{ti} is grand-mean-centered into TVx_{ti}, WITHOUT PMx_i at L2:

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(\mathbf{TVx_{ti}}) + \mathbf{e_{ti}}$

$TVx_{ti} = x_{ti} - C$ → it still has both Level-2 BP and Level-1 WP variation

Level 2: $\beta_{0i} = \gamma_{00} + U_{0i}$

$\beta_{1i} = \gamma_{10}$

γ_{10} = *smushed* WP and BP effects

Because TVx_{ti} still contains its original 2 different kinds of variation (BP and WP), its 1 fixed effect has to do the work of 2 predictors!

A *smushed* effect is also referred to as the *convergence*, *conflated*, or *composite* effect

Convergence (Smushed) Effect of a Time-Varying Predictor

$$\text{Convergence Effect: } \gamma_{\text{conv}} \approx \frac{\dfrac{\gamma_{\text{BP}}}{\text{SE}^2_{\text{BP}}} + \dfrac{\gamma_{\text{WP}}}{\text{SE}^2_{\text{WP}}}}{\dfrac{1}{\text{SE}^2_{\text{BP}}} + \dfrac{1}{\text{SE}^2_{\text{WP}}}}$$

Adapted from Raudenbush & Bryk (2002, p. 138)

- **The convergence effect will often be closer to the within-person effect** (due to larger level-1 sample size and thus smaller SE)
- **It is the rule, not the exception, that between and within effects differ** (Snijders & Bosker, 1999, p. 52-56, and personal experience!)
- However—when grand-mean-centering a time-varying predictor, **convergence is testable** by including a **contextual effect (carried by the person mean)** for how the **BP effect** differs from the **WP effect**…

Within-Person Fluctuation Model with **Grand-Mean-Centered Level-1 x_{ti}**

→ Model tests difference of WP vs. BP effects (It's been fixed!)

x_{ti} is grand-mean-centered into TVx_{ti}, <u>WITH</u> PMx_i at L2:

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(TVx_{ti}) + e_{ti}$

$TVx_{ti} = x_{ti} - C$ → it still has both Level-2 BP and Level-1 WP variation

Level 2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10}$

$PMx_i = \bar{X}_i - C$ → it has only Level-2 BP variation

γ_{10} becomes the WP effect → *unique* level-1 effect after controlling for PMx_i

γ_{01} *becomes* the contextual effect that indicates <u>how the BP effect differs from the WP effect</u> → *unique* level-2 effect after controlling for TVx_{ti} → does usual level matter beyond current level?

Person-MC and Grand-MC Models are Equivalent Given a Fixed Level-1 Main Effect Only

Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti} - PMx_i) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10}$

→ $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + e_{ti}$

→ $y_{ti} = \gamma_{00} + (\gamma_{01} - \gamma_{10})(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$

Composite Model:
← In terms of P-MC
← In terms of G-MC

Grand-MC: $TVx_{ti} = x_{ti}$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti}) + e_{ti}$

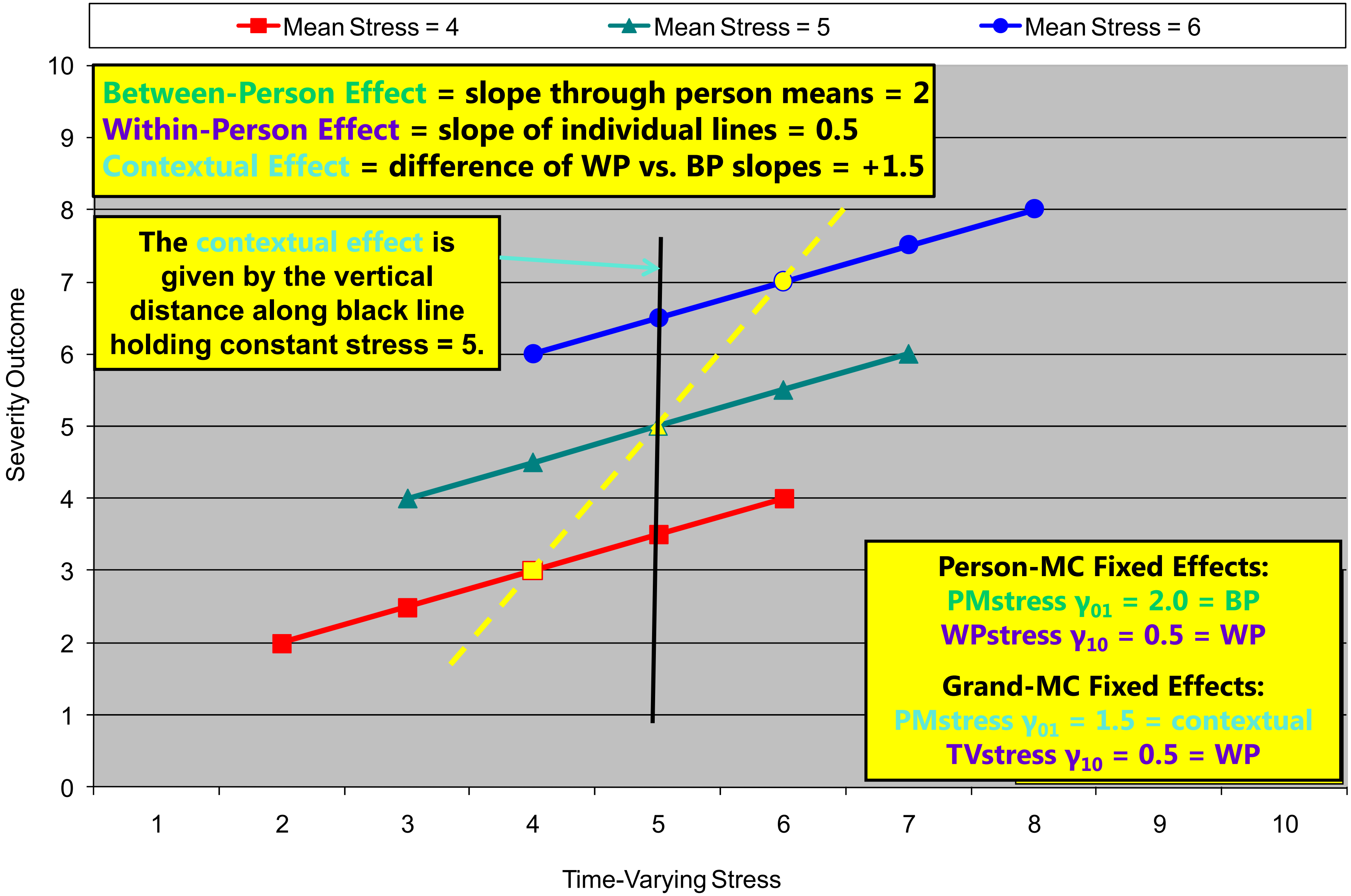
Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10}$

→ $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$

Effect	P-MC	G-MC
Intercept	γ_{00}	γ_{00}
WP Effect	γ_{10}	γ_{10}
Contextual	$\gamma_{01} - \gamma_{10}$	γ_{01}
BP Effect	γ_{01}	$\gamma_{01} + \gamma_{10}$

P-MC vs. G-MC: Interpretation Example



Summary: 3 Effects for TV Predictors

- **Is the Between-Person (BP) effect significant?**
  - Are people with higher predictor values than other people (*on average over time*) also higher on Y than other people (*on average over time*), such that the person mean of the TV predictor accounts for level-2 random intercept variance ($\tau^2_{U_0}$)?
  - Given directly by level-2 effect of $\mathrm{PMx_i}$ if using Person-MC for the level-1 predictor (or can be requested via ESTIMATE if using Grand-MC for the level-1 predictor)

- **Is the Within-Person (WP) effect significant?**
  - If you have higher predictor values than usual (*at this occasion*), do you also have higher outcomes values than usual (*at this occasion*), such that the within-person deviation of the TV predictor accounts for level-1 residual variance (σ^2_e)?
  - Given directly by the level-1 effect of $\mathrm{WPx_{ti}}$ if using Person-MC —*OR* — given directly by the level-1 effect of $\mathrm{TVx_{ti}}$ if using Grand-MC and including $\mathrm{PMx_i}$ at level 2 (without $\mathrm{PMx_i}$, the level-1 effect of $\mathrm{TVx_{ti}}$ if using Grand-MC is the smushed effect)

- **Are the BP and WP Effects different sizes: Is there a contextual effect?**
  - After controlling for the absolute value of TV predictor value at each occasion, is there still an incremental contribution from having a higher person mean of the TV predictor (i.e., does one's general tendency predict $\tau^2_{U_0}$ above and beyond)?
  - Given directly by level-2 effect of $\mathrm{PMx_i}$ if using Grand-MC for the level-1 predictor (or can be requested via ESTIMATE if using Person-MC for the level-1 predictor)

Variance Accounted For By Level-1 Predictors

- **Fixed effects of level 1 predictors *by themselves*:**
  - Level-1 (WP) main effects reduce Level-1 (WP) residual variance
  - Level-1 (WP) interactions also reduce Level-1 (WP) residual variance

- **What happens at level 2 depends on what kind of variance the level-1 predictor has:**
  - If the level-1 predictor ALSO has level-2 variance (e.g., Grand-MC predictors), then its level-2 variance will also likely reduce level-2 random intercept variance
  - If the level-1 predictor DOES NOT have level-2 variance (e.g., Person-MC predictors), then its reduction in the level-1 residual variance will cause an INCREASE in level-2 random intercept variance
    - Same thing happens with Grand-MC level-1 predictors, but you don't generally see it
  - It's just an artifact that the estimate of true random intercept variance is:

    True $\tau^2_{U_0}$ = observed $\tau^2_{U_0} - \frac{\sigma^2_e}{n}$ → so if only σ^2_e decreases, $\tau^2_{U_0}$ increases

The Joy of Interactions Involving Time-Varying Predictors

- <u>Must consider interactions with both its BP and WP parts:</u>
- Example: Does time-varying stress (x_{ti}) interact with sex (Sex_i)?

- <u>Person-Mean-Centering:</u>
  - $\mathbf{WPx_{ti}} * \mathbf{Sex_i}$ → Does the WP stress effect differ between men and women?
  - $\mathbf{PMx_i} * \mathbf{Sex_i}$ → Does the BP stress effect differ between men and women?
    - Not controlling for current levels of stress
    - If forgotten, then $\mathbf{Sex_i}$ moderates the stress effect only at level 1 (WP, not BP)

- <u>Grand-Mean-Centering:</u>
  - $\mathbf{TVx_{ti}} * \mathbf{Sex_i}$ → Does the WP stress effect differ between men and women?
  - $\mathbf{PMx_i} * \mathbf{Sex_i}$ → Does the *contextual* stress effect differ b/t men and women?
    - Incremental BP stress effect *after controlling for current levels of stress*
    - If forgotten, then although the level-1 main effect of stress has been un-smushed via the main effect of $\mathbf{PMx_i}$, the interaction of $\mathbf{TVx_{ti}} * \mathbf{Sex_i}$ would still be smushed

Interactions with Time-Varying Predictors: Example: TV Stress (x_{ti}) by Gender (Sex_i)

Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti} - PMx_i) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + \gamma_{11}(Sex_i)$

Composite: $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + e_{ti}$
$+ \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + \gamma_{11}(Sex_i)(x_{ti} - PMx_i)$

Grand-MC: $TVx_{ti} = x_{ti}$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti}) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + \gamma_{11}(Sex_i)$

Composite: $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$
$+ \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + \gamma_{11}(Sex_i)(x_{ti})$

Interactions Involving Time-Varying Predictors Belong at Both Levels of the Model

On the left below → Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

$y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + e_{ti}$
$+ \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + \gamma_{11}(Sex_i)(x_{ti} - PMx_i)$

← Composite model written as Person-MC

$y_{ti} = \gamma_{00} + (\gamma_{01} - \gamma_{10})(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$
$+ \gamma_{02}(Sex_i) + (\gamma_{03} - \gamma_{11})(Sex_i)(PMx_i) + \gamma_{11}(Sex_i)(x_{ti})$

← Composite model written as Grand-MC

On the right below → Grand-MC: $TVx_{ti} = x_{ti}$

$y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$
$+ \gamma_{02}(Sex_i) + \gamma_{03}(Sex_i)(PMx_i) + \gamma_{11}(Sex_i)(x_{ti})$

After adding an interaction for Sex_i with stress at both levels, then the Person-MC and Grand-MC models are equivalent

---

Intercept: $\gamma_{00} = \gamma_{00}$ **BP Effect:** $\gamma_{01} = \gamma_{01} + \gamma_{10}$ **Contextual:** $\gamma_{01} = \gamma_{01} - \gamma_{10}$

WP Effect: $\gamma_{10} = \gamma_{10}$ **BP*Sex Effect:** $\gamma_{03} = \gamma_{03} + \gamma_{11}$ **Contextual*Sex:** $\gamma_{03} = \gamma_{03} - \gamma_{11}$

Sex Effect: $\gamma_{20} = \gamma_{20}$ **BP*WP or Contextual*WP is the same:** $\gamma_{11} = \gamma_{11}$

Intra-variable Interactions

- Still must consider interactions with both its BP and WP parts!
- Example: Interaction of TV stress (x_{ti}) with person mean stress (PMx_i)

- Person-Mean-Centering:
  - $\mathbf{WPx_{ti}} * \mathbf{PMx_i}$ → Does the WP stress effect differ by overall stress level?
  - $\mathbf{PMx_i} * \mathbf{PMx_i}$ → Does the BP stress effect differ by overall stress level?
    - Not controlling for current levels of stress
    - If forgotten, then $\mathbf{PMx_i}$ moderates the stress effect only at level 1 (WP, not BP)

- Grand-Mean-Centering:
  - $\mathbf{TVx_{ti}} * \mathbf{PMx_i}$ → Does the WP stress effect differ by overall stress level?
  - $\mathbf{PMx_i} * \mathbf{PMx_i}$ → Does the *contextual* stress effect differ by overall stress?
    - Incremental BP stress effect *after controlling for current levels of stress*
    - If forgotten, then although the level-1 main effect of stress has been un-smushed via the main effect of $\mathbf{PMx_i}$, the interaction of $\mathbf{TVx_{ti}} * \mathbf{PMx_i}$ would still be smushed

Intra-variable Interactions:
Example: TV Stress (x_{ti}) by Person Mean Stress (PMx_i)

Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti} - PMx_i) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{02}(PMx_i)(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + \gamma_{11}(PMx_i)$

Composite: $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + e_{ti}$
$+ \gamma_{02}(PMx_i)(PMx_i) + \gamma_{11}(PMx_i)(x_{ti} - PMx_i)$

Grand-MC: $TVx_{ti} = x_{ti}$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti}) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{02}(PMx_i)(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + \gamma_{11}(PMx_i)$

Composite: $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti}$
$+ \gamma_{02}(PMx_i)(PMx_i) + \gamma_{11}(PMx_i)(x_{ti})$

Intra-variable Interactions:
Example: TV Stress (x_{ti}) by Person Mean Stress (PMx_i)

On the left below → Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

$$y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + e_{ti} + \gamma_{02}(PMx_i)(PMx_i) + \gamma_{11}(PMx_i)(x_{ti} - PMx_i)$$

← Written as Person-MC

$$y_{ti} = \gamma_{00} + (\gamma_{01} - \gamma_{10})(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti} + (\gamma_{02} - \gamma_{11})(PMx_i)(PMx_i) + \gamma_{11}(PMx_i)(x_{ti})$$

← Written as Grand-MC

On the right below → Grand-MC: $TVx_{ti} = x_{ti}$

$$y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + e_{ti} + \gamma_{02}(PMx_i)(PMx_i) + \gamma_{11}(PMx_i)(x_{ti})$$

After adding an interaction for PMx_i with stress at both levels, then the Person-MC and Grand-MC models are equivalent

---

Intercept: $\gamma_{00} = \gamma_{00}$ **BP Effect:** $\gamma_{01} = \gamma_{01} + \gamma_{10}$ **Contextual:** $\gamma_{01} = \gamma_{01} - \gamma_{10}$

WP Effect: $\gamma_{10} = \gamma_{10}$ **BP² Effect:** $\gamma_{02} = \gamma_{02} + \gamma_{11}$ **Contextual²:** $\gamma_{02} = \gamma_{02} - \gamma_{11}$

BP*WP or Contextual*WP is the same: $\gamma_{11} = \gamma_{11}$

When Person-MC ≠ Grand-MC: Random Effects of TV Predictors

Person-MC: $WPx_{ti} = x_{ti} - PMx_i$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti} - PMx_i) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + U_{1i}$

Variance due to PMx_i is removed from the random slope in Person-MC.

→ $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti} - PMx_i) + U_{0i} + U_{1i}(x_{ti} - PMx_i) + e_{ti}$

Grand-MC: $TVx_{ti} = x_{ti}$

Level-1: $y_{ti} = \beta_{0i} + \beta_{1i}(x_{ti}) + e_{ti}$

Level-2: $\beta_{0i} = \gamma_{00} + \gamma_{01}(PMx_i) + U_{0i}$

$\beta_{1i} = \gamma_{10} + U_{1i}$

Variance due to PMx_i is still part of the random slope in Grand-MC. So these models cannot be made equivalent.

→ $y_{ti} = \gamma_{00} + \gamma_{01}(PMx_i) + \gamma_{10}(x_{ti}) + U_{0i} + U_{1i}(x_{ti}) + e_{ti}$

Random Effects of TV Predictors

- **Random intercepts** mean different things under each model:
  - **Person-MC** → Individual differences at **WPx_{ti} =0** (that everyone has)
  - **Grand-MC** → Individual differences at **TVx_{ti}=0** (that not everyone has)

- **Differential shrinkage of the random intercepts** results from differential reliability of the intercept data across models:
  - Person-MC → Won't affect shrinkage of slopes unless highly correlated
  - Grand-MC → Will affect shrinkage of slopes due to forced extrapolation

- As a result, the **random slope variance may be too small** when using Grand-MC rather than Person-MC
  - Problem worsens with greater ICC of TV Predictor (more extrapolation)
  - Anecdotal example using clustered data was presented in Raudenbush & Bryk (2002; chapter 6)

Modeling Time-Varying <u>Categorical</u> Predictors

- Person-MC and Grand-MC really only apply to *continuous* TV predictors, but the need to consider BP and WP effects applies to *categorical* TV predictors too
- Binary level-1 predictors do not lend themselves to Person-MC
  - e.g., x_{ti} = 0 or 1 per occasion, person mean = .50 across occasions → impossible values
  - If x_{ti} = 0, then WPx_{ti} = 0 − .50 = − 0.50; If x_{ti} = 1, then WPx_{ti} = 1 − .50 = 0.50
  - Better: Leave x_{ti} uncentered and include person mean as level-2 predictor (results ~ Grand-MC)
- For >2 categories, person means of multiple dummy codes starts to break down, but we can think about types of people, and code BP effects accordingly
- <u>Example: Dementia present/not at each time point?</u>
  - **BP effects →** Ever diagnosed with dementia (no, yes)?
    - People who will eventually be diagnosed may differ prior to diagnosis (a BP effect)
  - **TV effect →** Diagnosed with dementia at each time point (no, yes)?
    - Acute differences of before/after diagnosis logically can only exist in the "ever" people
- Other examples: Mentor status, father absence, type of shift work (AM/PM)

Wrapping Up: Person-MC vs. Grand-MC

- Time-varying predictors carry at least two potential effects:
  - Some people are higher/lower than other people → BP, level-2 effect
  - Some occasions are higher/lower than usual → WP, level-1 effect

- BP and WP effects almost always need to be represented by two or more model parameters, using either:
  - *Person-mean-centering* (WPx_{ti} and PMx_i): WP ≠ 0?, BP ≠ 0?
  - *Grand-mean-centering* (TVx_{ti} and PMx_i): WP ≠ 0?, BP ≠ WP?
  - Both yield equivalent models if the level-1 WP effect is fixed, but not if the level-1 WP effect is random
    - Grand MC → *absolute* effect of x_{ti} varies randomly over people
    - Person MC → *relative* effect of x_{ti} varies randomly over people
    - Use prior theory and empirical data (ML AIC, BIC) to decide

Checking for Violations of Model Assumptions: Why should we care?

- "Fitting a model with untenable assumptions is as senseless as fitting a model to data that are knowingly flawed" (Singer & Willett, pg. 127)

- HOWEVER:
  - We don't actually know the true population relationships, so we don't know when our estimates, SE's, and *p*-values are off
  - Recommended strategy: "check assumptions of several initial models and any model you cite or interpret explicitly"
  - Mostly informal inspection – requires judgment call
    - "We prefer visual inspection of residual distributions" (S & W pg. 128)
  - Some things are fixable, some things are not
  - **End goal: Analyze the data the least wrong way possible** (because all models are wrong; some are useful)

General Consequences of Violating Model Assumptions

2 parts of the model to be concerned with:

- **Model for means = fixed effects**
  - Estimates depend on having the “right” model for the means → all relevant predictors, measured with as little error as possible
  - To the extent that predictors are missing or their effects are specified incorrectly, **fixed effect estimates will be biased**
- **Model for the variances = random effects and residuals**
  - SE and *p*-values of fixed effects depend on having the “right” model for the variances → most closely approximate actual data
  - To the extent that the model for the variances is off, **fixed effects SE and *p*-values will be off, too (biased)**
  - Because the general linear mixed model is estimated using a multivariate normal distribution for the **V** matrix, certain assumptions follow…

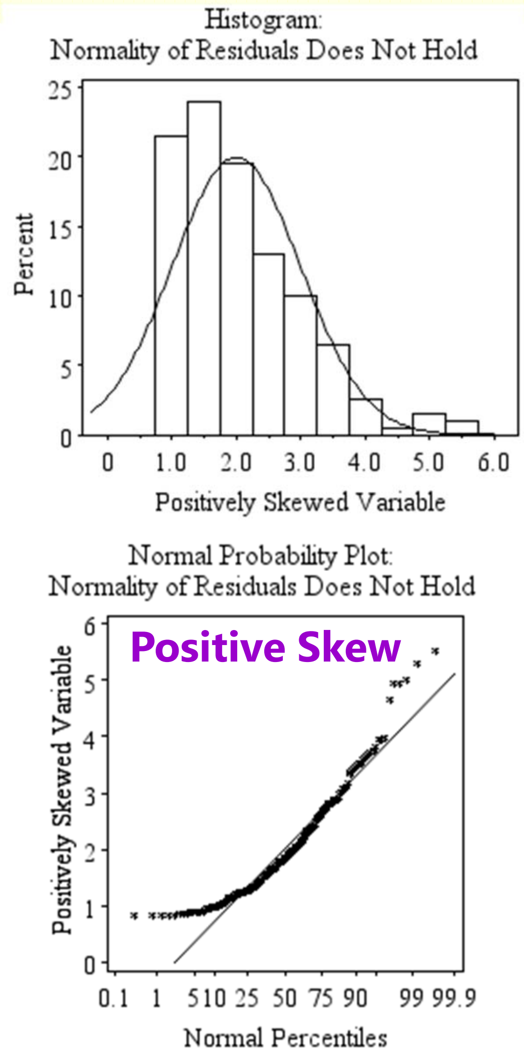
General Linear Mixed Model Assumptions

- GLM Assumptions:
  - Normality of **residuals** (not outcomes)
  - Independence and constant variance of **residuals**
    - Across sampling units
    - Across predictors

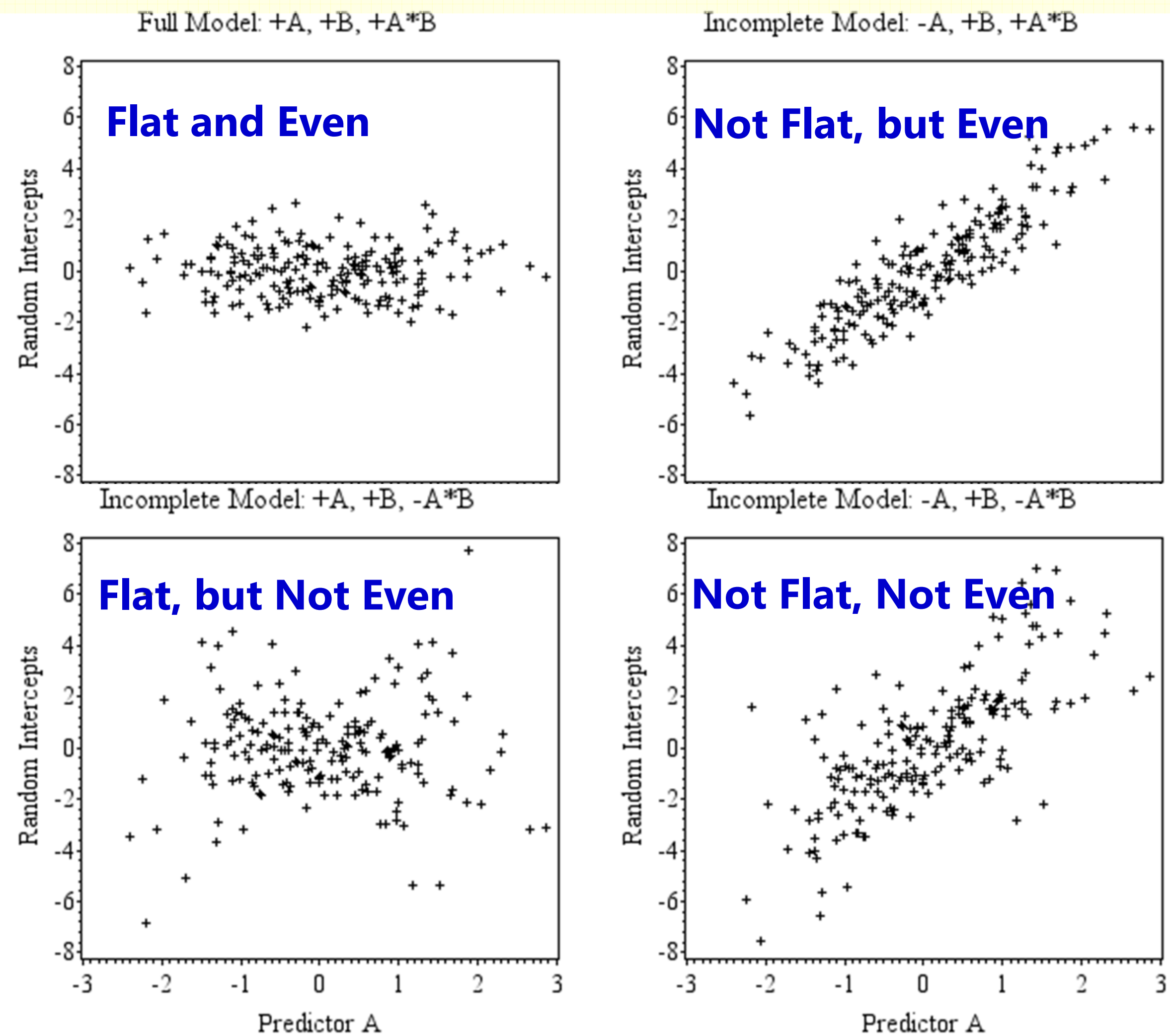
- MLM Assumptions are the same, except:
  - Apply **at** each level and **across** levels
  - More general options are available for changing the model to accommodate violations of assumptions if needed (goal is to **transform the model**, not the data)
  - ML also assumes MAR for any missing outcomes

Plots to Assess Assumptions:

Normality



Independence & Constant Variance



MLM Assumptions: **Normality** Multiple ‘residuals’ to consider:

<u>Level-1 e_{ti} residuals → (multivariate) normal distribution</u>

→ $e_{ti} \sim N(0, \mathbf{R})$ where $\mathbf{R} = \sigma_e^2$

→ e_{ti} has a mean = 0 and some estimated variance(s) and potentially covariances as well (is an empirical question)

<u>Level-2 **U**$_i$’s → multivariate normal distribution</u>

→ $U_{0i}, U_{1i}, \ldots \sim N(0, \mathbf{G})$

→ If random intercept: $G = \begin{pmatrix} \tau^2_{U0} & \\ & \end{pmatrix}$ If random slopes: $G = \begin{pmatrix} \tau^2_{U0} & \\ \tau_{U01} & \tau^2_{U1} \end{pmatrix}$

→ U’s EACH have a mean = 0 and some estimated variance, with estimated covariances between them

- The actual mean of U has another name: __________
- Covariances not included by default: added with TYPE=UN

3 Solutions for Non-Normality

1. <u>Pick a new model for the level-1 e_{ti} residuals</u>
   - **General*ized* linear mixed models** to the rescue!
     - Binary → Logit or Probit, Ordinal → Cumulative Logit
     - Count → Poisson or Negative Binomial (+ Zero-Inflated versions)
   - Unfortunately, level-2 **U**'s are still assumed multivariate normal
     - Problems with skewness → random effects CI's go out of bounds
   - Tricky to estimate, but should use ML with numeric integration when possible (try to avoid older "pseudo" or "quasi" ML options)

2. <u>Transform your data… carefully if at all…</u>
   - Assumptions apply to residuals, not to data!
   - Complicates interpretations (linear relationships → nonlinear)
   - Inherently subjective (especially "outlier" removal)
     - Check for extreme leverage on solution instead via INFLUENCE options after / on MODEL statement in PROC MIXED

3. Robust ML for Non-Normality

- **MLR in Mplus**: ≈ Yuan-Bentler T_2 (permits MCAR or MAR missing)
  - Same estimates and -2LL, corrected standard errors for all model parameters

- **χ^2-based fit statistics** are adjusted based on an estimated **scaling factor:**
  - Scaling factor = 1.000 = perfectly multivariate normal = same as ML
  - Scaling factor > 1.000 = leptokurtosis (too-fat tails; fixes too big χ^2)
  - Scaling factor < 1.000 = platykurtosis (too-thin tails; fixes too small χ^2)

- **SEs** computed with Huber-White 'sandwich' estimator → uses an information matrix from the variance of the partial first derivatives to correct the information matrix from the partial second derivatives
  - Leptokurtosis (too-fat tails) → increases information; fixes too small SEs
  - Platykurtosis (too-thin tails) → lowers information; fixes too big SEs

- In **SAS**: use "EMPIRICAL" option in PROC MIXED line
  - SEs are computed the same way but for fixed effects only, but can be unstable in unbalanced data, especially in small samples
  - SAS does not provide the needed scaling factor to adjust -2ΔLL test (not sure if this is a problem if you just use the fixed effect *p*-values)

Independence of Residuals At Level 1:

- Level-1 e_{ti} residuals are uncorrelated across level-1 units
  - Once random effects are modeled, residuals of the occasions from the same person are no longer correlated
- Solution for clustered or longitudinal models**:**
  - Choose the 'right' specification of random effects
    - Random effects go in **G**; **what's left in R is uncorrelated across observations**
- Another solution for longitudinal models:
  - Choose the 'right' alternative for the structure of the residual variances and covariances over time
  - Use **R** matrix or **G** and **R** matrices to better approximate observed data:
    - Are the residuals still correlated (AR1, TOEP) after random effects?
    - Are the variances over time homogeneous or heterogeneous?
      - This falls under the "constant variance" assumption – more on that later

Independence of Residuals At Level 2:

- <u>Level-2 $\mathbf{U}_i$'s are uncorrelated across level-2 units</u>
  - Implies no additional effects of clustering/nesting across persons after controlling for person-level predictors

- <u>Two alternatives to deal with additional clustering/nesting:</u>
  - Via fixed effects: Add dummy codes as level-2 predictors
    - Adjusts model for mean differences,
      but DOES NOT allow you to predict those mean differences
  - Via random effects: Add more levels (e.g., for family, group)
    - Adjusts model for mean differences,
      and it DOES allow you to predict those mean differences
    - Like adding another part to **G**

Independence of Residuals Across Levels:

- Level-1 e_{ti} residuals and Level-2 $\mathbf{U}_i$'s are also uncorrelated
  - Implies that what's left over at level-1 is not related to what's left over at level 2
  - Could be violated if level-2 effects are not modeled separately from level-1 effects (i.e., if convergence of level-1 predictors is assumed when it shouldn't be)

- Solution: Don't smush anything!
  - Allow different effects across upper levels for any lower-level predictor with respect to both main effects and interactions

Constant Variance of Residuals Across Sampling Units:

- **Level-2 U_i's** have constant variance across **level-2 units**
  - Implies no subgroups of individuals or groups that are more or less variable in terms of their distributions of random effects
  - If not, we can fit a heterogeneous variance model instead (stay tuned)
- **Level-1 e_{ti} residuals** have constant variance across **level-2 units***
  - Implies equal unexplained within-person variability across persons
  - Check for missing random effects of level-1 X's or cross-level interactions
  - If not, we can fit a heterogeneous variance model instead (stay tuned)
- **Level-1 e_{ti} residuals** have constant variance across **level-1 units**
  - Implies equal unexplained within-person variability across occasions
  - Can add additional random slopes for time or fit a heterogeneous variance model instead (e.g., TOEPH instead of TOEP, data permitting)
- * Test for heterogeneity of level-1 residuals applicable sometimes if *n* > 10 or so (see Snijders & Bosker, 1999, p. 126-7)

Independence and Constant Variance of Residuals Across Predictors:

- **Level-1 e_{ti} residuals** are flat with constant variance across **level-1 X's**
  - Implies no remaining relationship of X-Y **at level 1**
  - Specific example: level-1 residuals are flat and even across time after fixed and random effects (but we can fit separate variances by time if needed)
  - Check for potential nonlinear effects of level-1 predictors
- **Level-2 U_i's** are flat with constant variance across **level-1 X's**
  - Only possible relation between level-2 U_i and level-1 X is through relationship between level-2 PMx and level-2 U_i (so include PMx to avoid smushing)
- **Level-1 e_{ti} residuals** are flat with constant variance across **level-2 X's**
  - If not, we can fit a heterogeneous variance model instead (stay tuned)
- **Level-2 U_i's** are flat with constant variance across **level-2 X's**
  - Implies no remaining relationship of X-Y **at level 2**
  - Check for potential nonlinear effects of level-2 predictors
  - If not, we can fit a heterogeneous variance model instead (stay tuned)

Heterogeneous Variance Models

- Besides having random effects, predictors can play a role in predicting heterogeneity of variance at their level or lower:
  - Level-2 predictors → Differential level-2 random effects variances τ^2_U
    → Differential level-1 residual variances σ^2_e
  - Level-1 predictors → Differential level-1 residual variances σ^2_e
  - -2ΔLL tests used to see if extra heterogeneity effects are helpful

- Level-2 predictor of level-2 random effects variances for WP change:
  - e.g., changes in height over time in boys and in girls?
  - Boys may be taller and grow faster than girls on *average*
    - Effect of sex and sex*time → predict level of Y in **model for the means**
  - Boys may be more *variable* than girls in their levels and rates of change in height
    - Effect of sex → different τ^2_U in **level-2 model for the variances**

Heterogeneous Variance Models

- Level-2 predictor of level-2 and level-1 variances for WP fluctuation:
  - e.g., daily fluctuation in mood in men and in women
  - Men may have worse negative mood than women on *average*
    - Effect of sex → predict level of Y in **model for the means**
  - There may be greater *variability* among men than women in mean mood
    - Effect of sex → different τ_U^2 in **level-2 model for the variances**
  - Men may be more *variable* than women in their daily mood fluctuation
    - Effect of sex → different σ_e^2 in **level-1 model for the variances**

- Level-1 predictor of level-1 variance for WP fluctuation:
  - e.g., daily fluctuation in mood on stress/non-stress days
  - Negative mood may be worse on *average* on stress days than non-stress days
    - Effect of stress → predict level of Y in **model for the means**
  - There may be greater *variation* in mood on stress days than on non-stress days
    - Effect of stress → different σ_e^2 in **level-1 model for the variances**

Estimating Heterogeneous Variance Models via PROC MIXED

- Different variances via *GROUP=groupvar* option after the / on the RANDOM statement for level 2 or REPEATED statement for level 1
  - Less flexible than multiple-group SEM because the whole **G** and/or **R** matrix is either the same or different across groups (all or nothing)
  - GROUP= is limited to categorical predictors (must use CLASS statement)
    - Continuous level-2 predictors must use NLMIXED custom function instead

- In addition, different level-1 residual variances can be modeled via the LOCAL=EXP( ) option after / on REPEATED statement
  - For categorical or continuous level-2 or level-1 predictors
  - Cannot be used with any other **R** matrix structure besides VC
  - Predicts natural log of the residual variance so prediction can't go negative:

$$\sigma^2_{e_{ti}} = \alpha_0 \left(\exp\left[\alpha_1 X_1 + \alpha_2 X_2 \right] \right)$$

Estimating Heterogeneous Variance Models via PROC NLMIXED

- Can also write custom variance functions (see Hedeker's examples)
  - More flexible, linear models approach can accommodate any combination of categorical or continuous predictors
  - Here, an example of heterogeneous level-2 random intercept variance from Hoffman chapter 7 (see example for NLMIXED code)

Level 1:

$$\text{Symptoms}_{ti} = \beta_{0i} + e_{ti}$$

$$\text{Residual Variance: } \sigma^2_{e_{ti}} = \exp\left[\eta_{0i}\right]$$

Level 2:

$$\text{Intercept: } \beta_{0i} = \gamma_{00} + \gamma_{01}(\text{Women}_i) + \gamma_{02}(\text{Age}_i - 80) + \gamma_{03}(\text{Women}_i)(\text{Age}_i - 80) + U_{0i}$$

$$\text{Random Intercept Variance } \tau^2_{U_{0i}} = \exp\begin{bmatrix} \upsilon_{00} + \upsilon_{01}(\text{Women}_i) + \upsilon_{02}(\text{Age}_i - 80) \\ + \upsilon_{03}(\text{Women}_i)(\text{Age}_i - 80) \end{bmatrix}$$

$$\text{Residual Variance: } \eta_{0i} = \varepsilon_{00}$$

υ are effects for differential random intercept variance by intercept, sex, age and sex by age

η_{0i} is a placeholder (like β's in model for means)
ε_{00} is like fixed intercept of residual variance

Estimating Heterogeneous Variance Models via PROC NLMIXED

- Can test for a ω "scale factor"—like a random intercept for individual differences in residual variance (in WP variation)

From Hoffman chapter 7 (see example for NLMIXED code)

Level 1:

$$\text{Symptoms}_{ti} = \beta_{0i} + e_{ti}$$

$$\text{Residual Variance: } \sigma^2_{e_{ti}} = \exp\left[\eta_{0i}\right]$$

Level 2:

$$\text{Intercept: } \beta_{0i} = \gamma_{00} + \gamma_{01}(\text{Women}_i) + \gamma_{02}(\text{Age}_i - 80) + \gamma_{03}(\text{Women}_i)(\text{Age}_i - 80) + U_{0i}$$

$$\text{Random Intercept Variance } \tau^2_{U_{0i}} = \exp\left[\upsilon_{00}\right]$$

No υ predictors of differential random intercept variance, just an intercept here

$$\text{Residual Variance: } \eta_{0i} = \varepsilon_{00} + \omega_{0i}$$

η_{0i} is a placeholder (like β's in model for means)
ε_{00} is like fixed intercept of residual variance
ω_{0i} is like random intercept of residual variance

Estimating Heterogeneous Variance Models via PROC NLMIXED

Level 1:

$$\text{Symptoms}_{ti} = \beta_{0i} + \beta_{1i}\left(\text{Mood}_{ti} - \overline{\text{Mood}}_i\right) + \beta_{2i}\left(\text{Stressor}_{ti}\right) + e_{ti}$$

$$\text{Residual Variance: } \sigma^2_{e_{ti}} = \exp\left[\eta_{0i} + \underline{\eta_{1i}\left(\text{Mood}_{ti} - \overline{\text{Mood}}_i\right)} + \underline{\eta_{2i}\left(\text{Stressor}_{ti}\right)}\right]$$

Level 2:

$$\text{Intercept: } \beta_{0i} = \gamma_{00} + \gamma_{01}\left(\text{Women}_i\right) + \gamma_{02}\left(\text{Age}_i - 80\right) + \gamma_{03}\left(\text{Women}_i\right)\left(\text{Age}_i - 80\right)$$
$$+ \gamma_{04}\left(\overline{\text{Mood}}_i - 2\right) + \gamma_{08}\left(\overline{\text{Stressor}}_i - 0.40\right) + \gamma_{09}\left(\text{Women}_i\right)\left(\overline{\text{Stressor}}_i - 0.40\right)$$
$$+ \gamma_{0,16}\left(\overline{\text{Mood}}_i - 2\right)^2 + U_{0i}$$

$$\text{Within-Person Mood: } \beta_{1i} = \gamma_{10} + \gamma_{14}\left(\overline{\text{Mood}}_i - 2\right)$$

$$\text{Within-Person Stressor: } \beta_{2i} = \gamma_{20} + \gamma_{21}\left(\text{Women}_i\right)$$

$$\text{Random Intercept Variance } \tau^2_{U_{0i}} = \exp\begin{bmatrix} \upsilon_{00} + \upsilon_{01}\left(\text{Women}_i\right) + \upsilon_{02}\left(\text{Age}_i - 80\right) \\ + \upsilon_{04}\left(\overline{\text{Mood}}_i - 2\right) + \upsilon_{08}\left(\overline{\text{Stressor}}_i - 0.40\right) \end{bmatrix}$$

Residual Variance:

$$\eta_{0i} = \varepsilon_{00} + \underline{\varepsilon_{01}\left(\text{Women}_i\right)} + \underline{\varepsilon_{02}\left(\text{Age}_i - 80\right)} + \underline{\varepsilon_{04}\left(\overline{\text{Mood}}_i - 2\right)} + \underline{\varepsilon_{08}\left(\overline{\text{Stressor}}_i - 0.40\right)}$$

$$\eta_{1i} = \varepsilon_{10}$$

$$\eta_{2i} = \varepsilon_{20}$$

From Hoffman chapter 8 (see example for NLMIXED code)

υ predictors of differential random intercept variance

ε are predictors of differential residual variance
ω_{0i} was not estimable, so was not included

Assumptions of MLM: Summary

- Because model estimates, SEs, and fit statistics are derived from likelihood estimation using the multivariate normal distribution, their accuracy depends on its assumptions being met:
  - Residuals at each level (level 1 = e_{ti} values, level 2 = $\mathbf{U}_i$ values) are

    (1) normally distributed,

    (2) uncorrelated at each level and across levels,
    ($\mathbf{U}_i$ values can be correlated within their level), and

    (3) equally distributed across X's at each level and across levels.

- If not:

  (1) transform the data (meh) or pick a generalized model for non-linear outcomes (better when possible), or use robust ML for corrected SE's

  (2) add fixed or random effects (or a correlation over time),

  (3) make sure predictive relationships are correctly specified, and then consider heterogeneous variance models if needed.