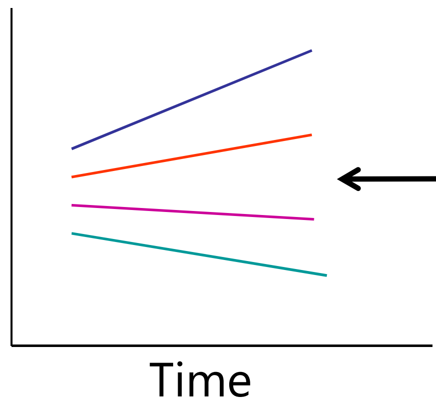
Describing Within-Person Fluctuation over Time using Alternative Covariance Structures

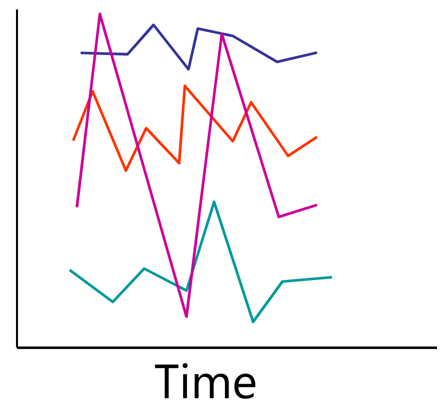
- Today's Class:
  - The Big Picture
  - ACS models using the **R** matrix only
  - Introducing the **G**, **Z**, and **V** matrices
  - ACS models combining the **G** and **R** matrices

Modeling Change vs. Fluctuation

Pure WP Change



Our focus this week → Pure WP Fluctuation



Model for the Means:

- WP Change → describe pattern of *average* change (over "time")
- **WP Fluctuation** → *may* not need anything (if no systematic change)

Model for the Variances:

- WP Change → describe *individual differences* in change (random effects)
  → this allows variances and covariances to differ over time
- **WP Fluctuation** → describe pattern of variances and covariances over time

Big Picture Framework: Models for the Variance in Longitudinal Data

Compound Symmetry (CS)

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

Most useful model: likely somewhere in between!

Unstructured (UN)

$$\begin{bmatrix} \sigma^2_{T1} & \sigma_{T12} & \sigma_{T13} & \sigma_{T14} \\ \sigma_{T21} & \sigma^2_{T2} & \sigma_{T23} & \sigma_{T24} \\ \sigma_{T31} & \sigma_{T32} & \sigma^2_{T3} & \sigma_{T43} \\ \sigma_{T41} & \sigma_{T42} & \sigma_{T43} & \sigma^2_{T4} \end{bmatrix}$$

Parsimony ⟵⟶ **Good fit**

Univariate RM ANOVA

NAME ...THAT ... STRUCTURE!

Multivariate RM ANOVA

What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Alternative Covariance Structure Models

- Useful in predicting patterns of variance and covariance that arise from fluctuation in the outcome across time:
  - **Variances**: Same (homogeneous) or different (heterogeneous)?
  - **Covariances**: Same or different? If different, what is the pattern?
    - Models with heterogeneous variances predict correlation instead of covariance
  - Often don't need any fixed effects for systematic effects of time in the model for the means (although this is always an empirical question)

- Limitations for most of the ACS models:
  - Require **equal-interval** occasions (they are based on idea of "time lag")
  - Require **balanced** time across persons (no intermediate time values)
  - But **do not require complete data** (unlike when CS and UN are estimated via least squares in ANOVA instead of ML/REML in MLM)

- ACS models do require some new terminology to introduce…

Two Families of ACS Models

- So far, we've referred to the variance and covariance matrix of the multivariate (longitudinal) outcomes as the **R** matrix
  - We now refer to these as **"R-only models"** (use **REPEATED** statement only)
  - Although the **R** matrix is actually specified per individual, ACS models usually assume the same **R** matrix for everyone
  - **R** matrix is symmetric with dimensions *n* x *n*, in which *n* = # occasions per person (although people can have missing data, the same set of *possible* occasions is required across people to use most **R**-only models)

- **3 other matrices we'll see in "G and R combined" ACS models:**
  - **G** = matrix of random effects variances and covariances (stay tuned)
  - **Z** = matrix of values for predictors that have random effects (stay tuned)
  - **V** = symmetric *n* x *n* matrix of **total** variance and covariance over time
    - If the model includes random effects, then **G** and **Z** get combined with **R** to make **V** as $\mathbf{V} = \mathbf{ZGZ}^{\mathrm{T}} + \mathbf{R}$ (accomplished by adding the **RANDOM** statement)
    - If the model does NOT include random effects in **G**, then $\mathbf{V} = \mathbf{R}$… *so, **R**-only*

Review: Covariances and Correlations

$$\text{Correlation}_{y1,y2} = \frac{\text{Covariance}_{y1,y2}}{\sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}}$$

$$\text{Covariance}_{y1,y2} = \text{Correlation}_{y1,y2} * \sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}$$

- Given the standard deviation (as $\sqrt{\text{Variance}}$) at each occasion, either the correlation and covariance can be calculated given the other
- ACS models with **homogeneous variances** tend to be specified in terms of **variance and covariance**
  - Given same variance over time, same covariance → same correlation
- ACS models with **heterogeneous variance** tend to be specified in terms of **variance and correlation**
  - Different variances over time → different covariances over time, <u>even if the correlation is the same</u> (so only correlation is estimated directly)

R-Only ACS Models

- The **R-only** models to be presented next are all specified using the **REPEATED** statement only (no RANDOM statement)

- They are explained by showing their predicted **R** matrix, which provides the **total** variances and covariances across occasions
  - Total variance per occasion on diagonal
  - Total covariances across occasions on off-diagonals
  - I've included in " " the labels SAS uses for each parameter

- Correlations across occasions can be calculated given variances and covariances, which would be shown in the **RCORR** matrix (available in SAS PROC MIXED)
  - 1's on diagonal (standardized variables), correlations on off-diagonal

- **Unstructured (TYPE=UN) will always fit best by −2LL**
  - All ACS models are nested within Unstructured (UN = the data)
  - Goal: find an ACS model that is **simpler** but **not worse fitting** than UN

R-Only ACS Models: CS/CSH

- **Compound Symmetry: TYPE=CS**
  - 2 parameters:
    - **1 "residual" variance σ_e^2**
    - **1 "CS" covariance** across occasions
  - Constant total variance: $\text{CS} + \sigma_e^2$
  - Constant total covariance: CS

$$\begin{bmatrix} CS+\sigma_e^2 & CS & CS & CS \\ CS & CS+\sigma_e^2 & CS & CS \\ CS & CS & CS+\sigma_e^2 & CS \\ CS & CS & CS & CS+\sigma_e^2 \end{bmatrix}$$

- **Compound Symmetry Heterogeneous: TYPE=CSH**
  - *n*+1 parameters:
    - ***n* separate "Var(*n*)" total variances σ_{Tn}^2**
    - **1 "CSH" total correlation across occasions**
  - Separate total variances are estimated directly
  - Still constant total correlation: CSH (but has non-constant covariances)

$$\begin{bmatrix} \sigma_{T1}^2 & CSH\sigma_{T1}\sigma_{T2} & CSH\sigma_{T1}\sigma_{T3} & CSH\sigma_{T1}\sigma_{T4} \\ CSH\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & CSH\sigma_{T2}\sigma_{T3} & CSH\sigma_{T2}\sigma_{T4} \\ CSH\sigma_{T3}\sigma_{T1} & CSH\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & CSH\sigma_{T3}\sigma_{T4} \\ CSH\sigma_{T4}\sigma_{T1} & CSH\sigma_{T4}\sigma_{T2} & CSH\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

R-Only ACS Models: AR1/ARH1

- **1st Order Auto-Regressive: TYPE=AR(1)**
  - 2 parameters:
    - **1 constant total variance σ_T^2 (mislabeled "residual")**
    - **1 "AR1" total auto-correlation r_T** across occasions
    - r_T^1 is lag-1 correlation, r_T^2 is lag-2 correlation, r_T^3 is lag-3 correlation….

$$\begin{bmatrix} \sigma_T^2 & r_T^1\sigma_T^2 & r_T^2\sigma_T^2 & r_T^3\sigma_T^2 \\ r_T^1\sigma_T^2 & \sigma_T^2 & r_T^1\sigma_T^2 & r_T^2\sigma_T^2 \\ r_T^2\sigma_T^2 & r_T^1\sigma_T^2 & \sigma_T^2 & r_T^1\sigma_T^2 \\ r_T^3\sigma_T^2 & r_T^2\sigma_T^2 & r_T^1\sigma_T^2 & \sigma_T^2 \end{bmatrix}$$

- **1st Order Auto-Regressive Heterogeneous: TYPE=ARH(1)**
  - *n*+1 parameters:
    - ***n* separate "Var(*n*)" total variances σ_{Tn}^2**
    - **1 "ARH1" total auto-correlation r_T** across occasions
    - r_T^1 is lag-1 correlation, r_T^2 is lag-2 correlation, r_T^3 is lag-3 correlation….

$$\begin{bmatrix} \sigma_{T1}^2 & r_T^1\sigma_{T1}\sigma_{T2} & r_T^2\sigma_{T1}\sigma_{T3} & r_T^3\sigma_{T1}\sigma_{T4} \\ r_T^1\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & r_T^1\sigma_{T2}\sigma_{T3} & r_T^2\sigma_{T2}\sigma_{T4} \\ r_T^2\sigma_{T3}\sigma_{T1} & r_T^1\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & r_T^1\sigma_{T3}\sigma_{T4} \\ r_T^3\sigma_{T4}\sigma_{T1} & r_T^2\sigma_{T4}\sigma_{T2} & r_T^1\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

R-Only ACS Models: TOEP*n*/TOEPH*n*

- **Toeplitz(*n*): TYPE=TOEP(*n*)**
  - *n* parameters:
    - **1 constant total variance σ^2_T (mislabeled "residual")**
    - ***n*−1 "TOEP(*lag*)" c_{Tn} banded total covariances** across occasions
    - c_{T1} is lag-1 covariance, c_{T2} is lag-2 covariance, c_{T3} is lag-3 covariance….

$$\begin{bmatrix} \sigma^2_T & & & \\ c_{T1} & \sigma^2_T & & \\ c_{T2} & c_{T1} & \sigma^2_T & \\ c_{T3} & c_{T2} & c_{T1} & \sigma^2_T \end{bmatrix}$$

- **Toeplitz Heterogeneous(*n*): TYPE=TOEPH(*n*)**
  - *n* + (*n*−1) parameters:
    - ***n* separate "Var(*n*)" total variances σ^2_{Tn}**
    - ***n*−1 "TOEPH(*lag*)" r_{Tn} banded total correlations** across occasions
    - r_{T1} is lag-1 correlation, r_{T2} is lag-2 correlation, r_{T3} is lag-3 correlation….

$$\begin{bmatrix} \sigma^2_{T1} & r_{T1}\sigma_{T1}\sigma_{T2} & r_{T2}\sigma_{T1}\sigma_{T3} & r_{T3}\sigma_{T1}\sigma_{T4} \\ r_{T1}\sigma_{T2}\sigma_{T1} & \sigma^2_{T2} & r_{T1}\sigma_{T2}\sigma_{T3} & r_{T2}\sigma_{T2}\sigma_{T4} \\ r_{T2}\sigma_{T3}\sigma_{T1} & r_{T1}\sigma_{T3}\sigma_{T2} & \sigma^2_{T3} & r_{T1}\sigma_{T3}\sigma_{T4} \\ r_{T3}\sigma_{T4}\sigma_{T1} & r_{T2}\sigma_{T4}\sigma_{T2} & r_{T1}\sigma_{T4}\sigma_{T3} & \sigma^2_{T4} \end{bmatrix}$$

Comparing **R**-only ACS Models

- Baseline models: **CS =simplest**, **UN = most complex**
  - Relative to CS, more complex models fit "better" or "not better"
  - Relative to UN, less complex models fit "worse" or "not worse"

- Other rules of nesting and model comparisons:
  - Homogeneous variance models are nested within heterogeneous variance models (e.g., CS in CSH, AR1 in ARH1, TOEP in TOEPH)
  - CS and AR1 are each nested within TOEP (i.e., TOEP can become CS or AR1 through restrictions of its covariance patterns)
  - CS and AR1 are not nested (because both have 2 parameters)
  - **R**-only models differ in unbounded parameters, so can be compared using regular $-2\Delta LL$ tests (instead of mixture $-2\Delta LL$ tests)
  - Good idea to start by assuming heterogeneous variances until you settle on the covariance pattern, then test if het. var. are still necessary
  - When in doubt, just compare AIC and BIC (useful even with $-2\Delta LL$ tests)

The Other Family of ACS Models

- **R**-only models *directly* predict the **total** variance and covariance
- **G** and **R** models *indirectly* predict the total variance and covariance through **between-person (BP)** and **within-person (WP)** sources of variance and covariance → So, for this model: $\mathbf{y_{ti} = \beta_0 + U_{0i} + e_{ti}}$
  - **BP** = **G** matrix of **level-2 random effect ($\mathbf{U_{0i}}$)** variances and covariances
    - Which effects get to be random (whose variance and covariances are then included in **G**) is specified using the **RANDOM** statement (always TYPE=UN)
    - Our ACS models have a random intercept only, so **G** is 1x1 scalar of $[\boldsymbol{\tau}^2_{\mathbf{U_0}}]$
  - **WP** = **R** matrix of **level-1 ($\mathbf{e_{ti}}$) residual** variances and covariances
    - The *n* x *n* **R** matrix of **residual** variances and covariances **that remain** after controlling for random intercept variance is then modeled with **REPEATED**
  - **Total** = **V** = *n* x *n* matrix of **total** variance and covariance over time that results from putting **G** and **R** together: $\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}^{\mathrm{T}} + \mathbf{R}$
    - **Z** is a matrix that holds the values of predictors with random effects, but **Z** will be an *n* x 1 column of 1's for now (random intercept only)

A "Random Intercept" (**G** and **R**) Model

Total Predicted Data Matrix is called V Matrix

$$\begin{bmatrix} \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} + \sigma^2_e \end{bmatrix}$$

Level 2, BP Variance

Unstructured **G Matrix**
(RANDOM statement)
Each person has same **1 x 1 G** matrix (no covariance across persons in two-level model)

Level 1, WP Variance

Diagonal (VC) **R Matrix**
(REPEATED statement)
Each person has same ***n* x *n* R** matrix → **equal variances and 0 covariances** across time (no covariance across persons)

Random Intercept Variance only

$$\begin{bmatrix} \tau^2_{U_0} \end{bmatrix}$$

Residual Variance only

$$\begin{bmatrix} \sigma^2_e & 0 & 0 & 0 \\ 0 & \sigma^2_e & 0 & 0 \\ 0 & 0 & \sigma^2_e & 0 \\ 0 & 0 & 0 & \sigma^2_e \end{bmatrix}$$

CS as a "Random Intercept" Model

<u>RI and DIAG</u>: Total predicted data matrix is called V matrix, created from the G [TYPE=UN] and R [TYPE=VC] matrices as follows:

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_e & 0 & 0 & 0\\ 0 & \sigma^2_e & 0 & 0\\ 0 & 0 & \sigma^2_e & 0\\ 0 & 0 & 0 & \sigma^2_e\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e\end{bmatrix}$$

Does the end result V look familiar? It should: CS = τ^2_{U0}

$$\begin{bmatrix}CS+\sigma^2_e & CS & CS & CS\\ CS & CS+\sigma^2_e & CS & CS\\ CS & CS & CS+\sigma^2_e & CS\\ CS & CS & CS & CS+\sigma^2_e\end{bmatrix}$$

So if the **R-only CS model** (the simplest baseline) can be specified equivalently using **G and R**, can we do the same for the **R-only UN model** (the most complex baseline)?

Absolutely! ...*with one small catch*

UN via a "Random Intercept" Model

RI and UN*n*−1: Total predicted data matrix is called V matrix, created from the G [TYPE=UN] and R [TYPE=UN(*n*−1)] matrices as follows:

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau^2_{U_0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma^2_{e1} & \sigma_{e12} & \sigma_{e13} & 0 \\ \sigma_{e21} & \sigma^2_{e2} & \sigma_{e23} & \sigma_{e24} \\ \sigma_{e31} & \sigma_{e32} & \sigma^2_{e3} & \sigma_{e34} \\ 0 & \sigma_{e42} & \sigma_{e43} & \sigma^2_{e4} \end{bmatrix} = \begin{bmatrix} \tau^2_{U_0} + \sigma^2_{e1} & \tau^2_{U_0} + \sigma_{e12} & \tau^2_{U_0} + \sigma_{e13} & \tau^2_{U_0} \\ \tau^2_{U_0} + \sigma_{e21} & \tau^2_{U_0} + \sigma^2_{e2} & \tau^2_{U_0} + \sigma_{e23} & \tau^2_{U_0} + \sigma_{e24} \\ \tau^2_{U_0} + \sigma_{e31} & \tau^2_{U_0} + \sigma_{e32} & \tau^2_{U_0} + \sigma^2_{e3} & \tau^2_{U_0} + \sigma_{e34} \\ \tau^2_{U_0} & \tau^2_{U_0} + \sigma_{e42} & \tau^2_{U_0} + \sigma_{e43} & \tau^2_{U_0} + \sigma^2_{e4} \end{bmatrix}$$

This **RI and UN*n*−1 model** is equivalent to (makes same predictions as) the **R-only UN model**. But it shows the *residual* (not total) covariances.

Because we can't estimate all possible variances and covariances in the **R** matrix and also estimate the random intercept variance $\tau^2_{U_0}$ in the **G** matrix, we have to eliminate the last **R** matrix covariance by setting it to 0.

Accordingly, in the **RI and UN*n*−1** model, the random intercept variance $\tau^2_{U_0}$ takes on the value of the covariance for the first and last occasions.

Rationale for **G** and **R** ACS models

- Modeling WP fluctuation traditionally involves using **R** only (no **G**) → **Total** BP + WP variance described by just **R** matrix (so **R**=**V**)
  - Correlations would still be expected even at distant time lags because of constant individual differences (i.e., the BP random intercept)
  - Resulting **R**-only model may require lots of estimated parameters as a result e.g., 8 time points? Pry need a 7-lag Toeplitz(8) model

- **Why not take out the primary reason for the covariance across occasions (the random intercept variance) and see what's left?**
  - Random intercept variance $\boldsymbol{\tau}^2_{\mathbf{U_0}}$ in **G** → control for person mean differences
  - THEN predict just the **residual** variance/covariance in **R**, not the **total**
  - Resulting model may be more parsimonious (e.g., maybe only lag1 or lag2 occasions are still related after removing $\boldsymbol{\tau}^2_{\mathbf{U_0}}$ as a source of covariance)
  - Has the advantage of still distinguishing BP from WP variance (useful for descriptive purposes and for calculating effect sizes later)

Random Intercept + Diagonal **R** Models

RI and DIAG: V is created from **G [TYPE=UN] and R [TYPE=VC]:**
homogeneous *residual variances;* ***no*** *residual covariances*

Same fit as R-only CS

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_e & 0 & 0 & 0\\ 0 & \sigma^2_e & 0 & 0\\ 0 & 0 & \sigma^2_e & 0\\ 0 & 0 & 0 & \sigma^2_e\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e\end{bmatrix}$$

RI and DIAGH: V is created from **G [TYPE=UN] and R [TYPE=UN(1)]:**
heterogeneous *residual variances;* ***no*** *residual covariances*

NOT same fit as R-only CSH

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_{e1} & 0 & 0 & 0\\ 0 & \sigma^2_{e2} & 0 & 0\\ 0 & 0 & \sigma^2_{e3} & 0\\ 0 & 0 & 0 & \sigma^2_{e4}\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_{e1} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_{e2} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_{e3} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_{e4}\end{bmatrix}$$

Random Intercept + AR1 **R** Models

<u>RI and AR1</u>: V is created from **G [TYPE=UN] and R [TYPE=AR(1)]:**
homogeneous *residual variances;* ***auto-regressive lagged*** *residual covariances*

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_e & r^1_e\sigma^2_e & r^2_e\sigma^2_e & r^3_e\sigma^2_e\\ r^1_e\sigma^2_e & \sigma^2_e & r^1_e\sigma^2_e & r^2_e\sigma^2_e\\ r^2_e\sigma^2_e & r^1_e\sigma^2_e & \sigma^2_e & r^1_e\sigma^2_e\\ r^3_e\sigma^2_e & r^2_e\sigma^2_e & r^1_e\sigma^2_e & \sigma^2_e\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+r^1_e\sigma^2_e & \tau^2_{U_0}+r^2_e\sigma^2_e & \tau^2_{U_0}+r^3_e\sigma^2_e\\ \tau^2_{U_0}+r^1_e\sigma^2_e & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+r^1_e\sigma^2_e & \tau^2_{U_0}+r^2_e\sigma^2_e\\ \tau^2_{U_0}+r^2_e\sigma^2_e & \tau^2_{U_0}+r^1_e\sigma^2_e & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+r^1_e\sigma^2_e\\ \tau^2_{U_0}+r^3_e\sigma^2_e & \tau^2_{U_0}+r^2_e\sigma^2_e & \tau^2_{U_0}+r^1_e\sigma^2_e & \tau^2_{U_0}+\sigma^2_e\end{bmatrix}$$

<u>RI and ARH1</u>: V is created from **G [TYPE=UN] and R [TYPE=ARH(1)]:**
heterogeneous *residual variances;* ***auto-regressive lagged*** *residual covariances*

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_{e1} & r^1_e\sigma_{e1}\sigma_{e2} & r^2_e\sigma_{e1}\sigma_{e3} & r^3_e\sigma_{e1}\sigma_{e4}\\ r^1_e\sigma_{e2}\sigma_{e1} & \sigma^2_{e2} & r^1_e\sigma_{e2}\sigma_{e3} & r^2_e\sigma_{e2}\sigma_{e4}\\ r^2_e\sigma_{e3}\sigma_{e1} & r^1_e\sigma_{e3}\sigma_{e2} & \sigma^2_{e3} & r^1_e\sigma_{e3}\sigma_{e4}\\ r^3_e\sigma_{e4}\sigma_{e1} & r^2_e\sigma_{e4}\sigma_{e2} & r^1_e\sigma_{e4}\sigma_{e3} & \sigma^2_{e4}\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_{e1} & \tau^2_{U_0}+r^1_e\sigma_{e1}\sigma_{e2} & \tau^2_{U_0}+r^2_e\sigma_{e1}\sigma_{e3} & \tau^2_{U_0}+r^3_e\sigma_{e1}\sigma_{e4}\\ \tau^2_{U_0}+r^1_e\sigma_{e2}\sigma_{e1} & \tau^2_{U_0}+\sigma^2_{e2} & \tau^2_{U_0}+r^1_e\sigma_{e2}\sigma_{e3} & \tau^2_{U_0}+r^2_e\sigma_{e2}\sigma_{e4}\\ \tau^2_{U_0}+r^2_e\sigma_{e3}\sigma_{e1} & \tau^2_{U_0}+r^1_e\sigma_{e3}\sigma_{e2} & \tau^2_{U_0}+\sigma^2_{e3} & \tau^2_{U_0}+r^1_e\sigma_{e3}\sigma_{e4}\\ \tau^2_{U_0}+r^3_e\sigma_{e4}\sigma_{e1} & \tau^2_{U_0}+r^2_e\sigma_{e4}\sigma_{e2} & \tau^2_{U_0}+r^1_e\sigma_{e4}\sigma_{e3} & \tau^2_{U_0}+\sigma^2_{e4}\end{bmatrix}$$

Random Intercept + TOEP*n*−*1* **R** Models

<u>RI and TOEP*n*−*1*</u>: V is created from **G [TYPE=UN] and R [TYPE=TOEP(*n*−*1*)]:**
homogeneous *residual variances;* ***banded*** *residual covariances*

Same fit as R-only TOEP(*n*)

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau^2_{U_0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma^2_e & c_{e1} & c_{e2} & 0 \\ c_{e1} & \sigma^2_e & c_{e1} & c_{e2} \\ c_{e2} & c_{e1} & \sigma^2_e & c_{e1} \\ 0 & c_{e2} & c_{e1} & \sigma^2_e \end{bmatrix} = \begin{bmatrix} \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} + c_{e1} & \tau^2_{U_0} + c_{e2} & \tau^2_{U_0} \\ \tau^2_{U_0} + c_{e1} & \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} + c_{e1} & \tau^2_{U_0} + c_{e2} \\ \tau^2_{U_0} + c_{e2} & \tau^2_{U_0} + c_{e1} & \tau^2_{U_0} + \sigma^2_e & \tau^2_{U_0} + c_{e1} \\ \tau^2_{U_0} & \tau^2_{U_0} + c_{e2} & \tau^2_{U_0} + c_{e1} & \tau^2_{U_0} + \sigma^2_e \end{bmatrix}$$

Because of τ^2_{U0}, highest lag covariance in **R** must be set to 0 for model to be identified

<u>RI and TOEPH*n*−*1*</u>: V is created from **G [TYPE=UN] and R [TYPE=TOEPH(*n*−*1*)]:**
homogeneous *residual variances;* ***banded*** *residual covariances*

NOT same fit as R-only TOEPH(*n*)

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau^2_{U_0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma^2_{e1} & r_{e1}\sigma_{e1}\sigma_{e2} & r_{e2}\sigma_{e1}\sigma_{e3} & 0 \\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma^2_{e2} & r_{e1}\sigma_{e2}\sigma_{e3} & r_{e2}\sigma_{e2}\sigma_{e4} \\ r_{e2}\sigma_{e3}\sigma_{e1} & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma^2_{e3} & r_{e1}\sigma_{e3}\sigma_{e4} \\ 0 & r_{e2}\sigma_{e4}\sigma_{e2} & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma^2_{e4} \end{bmatrix} = \begin{bmatrix} \tau^2_{U_0} + \sigma^2_{e1} & \tau^2_{U_0} + r_{e1}\sigma_{e1}\sigma_{e2} & \tau^2_{U_0} + r_{e2}\sigma_{e1}\sigma_{e3} & \tau^2_{U_0} \\ \tau^2_{U_0} + r_{e1}\sigma_{e2}\sigma_{e1} & \tau^2_{U_0} + \sigma^2_{e2} & \tau^2_{U_0} + r_{e1}\sigma_{e2}\sigma_{e3} & \tau^2_{U_0} + r_{e2}\sigma_{e2}\sigma_{e4} \\ \tau^2_{U_0} + r_{e2}\sigma_{e3}\sigma_{e1} & \tau^2_{U_0} + r_{e1}\sigma_{e3}\sigma_{e2} & \tau^2_{U_0} + \sigma^2_{e3} & \tau^2_{U_0} + r_{e1}\sigma_{e3}\sigma_{e4} \\ \tau^2_{U_0} & \tau^2_{U_0} + r_{e2}\sigma_{e4}\sigma_{e2} & \tau^2_{U_0} + r_{e1}\sigma_{e4}\sigma_{e3} & \tau^2_{U_0} + \sigma^2_{e4} \end{bmatrix}$$

Random Intercept + TOEP2 **R** Models

RI and TOEP*2*: **V** is created from **G [TYPE=UN] and R [TYPE=TOEP(*2*)]:**
homogeneous *residual variances;* ***banded*** *residual covariance at* ***lag1*** *only*

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_e & c_{e1} & 0 & 0\\ c_{e1} & \sigma^2_e & c_{e1} & 0\\ 0 & c_{e1} & \sigma^2_e & c_{e1}\\ 0 & 0 & c_{e1} & \sigma^2_e\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+c_{e1} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0}+c_{e1} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+c_{e1} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0}+c_{e1} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0}+c_{e1}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+c_{e1} & \tau^2_{U_0}+\sigma^2_e\end{bmatrix}$$

Now we can test the need for residual covariances at higher lags

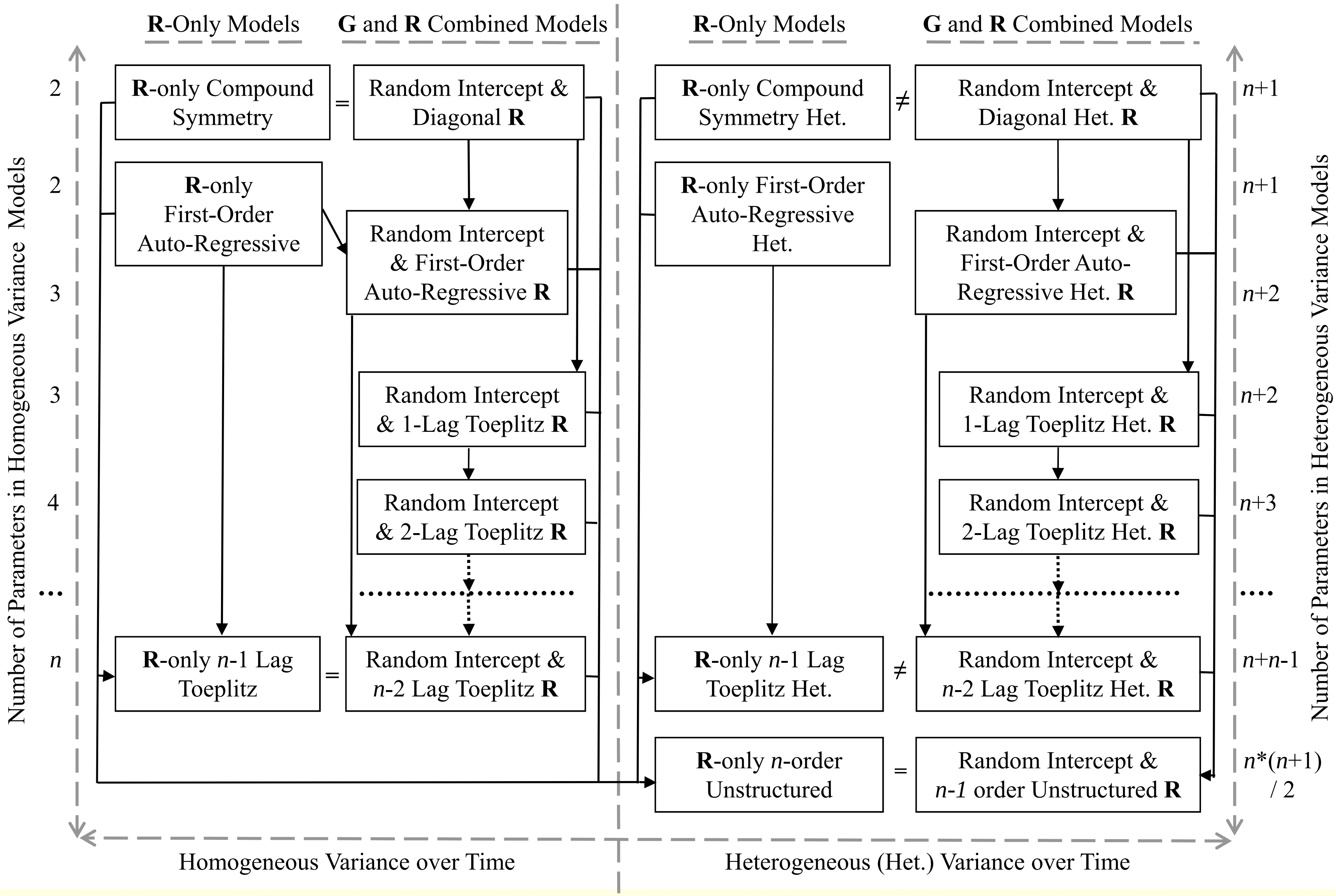
RI and TOEPH*1*: **V** is created from **G [TYPE=UN] and R [TYPE=TOEPH(*2*)]:**
homogeneous *residual variances;* ***banded*** *residual covariance at* ***lag1*** *only*

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}\tau^2_{U_0}\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} + \begin{bmatrix}\sigma^2_{e1} & r_{e1}\sigma_{e1}\sigma_{e2} & 0 & 0\\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma^2_{e2} & r_{e1}\sigma_{e2}\sigma_{e3} & 0\\ 0 & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma^2_{e3} & r_{e1}\sigma_{e3}\sigma_{e4}\\ 0 & 0 & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma^2_{e4}\end{bmatrix} = \begin{bmatrix}\tau^2_{U_0}+\sigma^2_{e1} & \tau^2_{U_0}+r_{e1}\sigma_{e1}\sigma_{e2} & \tau^2_{U_0} & \tau^2_{U_0}\\ \tau^2_{U_0}+r_{e1}\sigma_{e2}\sigma_{e1} & \tau^2_{U_0}+\sigma^2_{e2} & \tau^2_{U_0}+r_{e1}\sigma_{e2}\sigma_{e3} & \tau^2_{U_0}\\ \tau^2_{U_0} & \tau^2_{U_0}+r_{e1}\sigma_{e3}\sigma_{e2} & \tau^2_{U_0}+\sigma^2_{e3} & \tau^2_{U_0}+r_{e1}\sigma_{e3}\sigma_{e4}\\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+r_{e1}\sigma_{e4}\sigma_{e3} & \tau^2_{U_0}+\sigma^2_{e4}\end{bmatrix}$$

Map of **R**-only and **G** and **R** ACS Models

Arrows indicate nesting (end is more complex model)



Stuff to Watch Out For…

- **If using a random intercept, don't forget to drop 1 parameter in:**
  - ***n*-1 order UN R**: Can't get all possible elements in **R**, plus $\tau^2_{U_0}$ in **G**
  - **TOEP*n*−1**: Have to eliminate last lag covariance

- If using a random intercept…
  - Can't do RI + CS **R**: Can't get a constant in **R**, and then another constant in **G**
  - Can often test if random intercept helps (e.g., AR1 is nested within RI + AR1)

- If "**time**" is treated as **continuous** in the fixed effects, you will need another variable for **time** that is **categorical** to use in the syntax:
  - "Continuous Time" → on MODEL statement
  - "Categorical Time" → on CLASS and REPEATED statements

- Most alternative covariance structure models assume **time is balanced across persons with equal intervals across occasions**
  - If not, holding correlations of same lag equal doesn't make sense
  - Other structures can be used for unbalanced time
    - SP(POW)(time) = AR1 for unbalanced time (see SAS REPEATED statement for others)

Summary: Two Families of ACS Models

- **R**-only models:
  - Specify **R** model on REPEATED statement without any random effects variances in **G** (so no RANDOM statement is used)
  - Include UN, CS, CSH, AR1, AR1H, TOEP*n*, TOEPH*n* (among others)
  - *Total* variance and *total* covariance kept in **R**, so **R** = **V**
  - Other than CS, does not partition total variance into BP vs. WP

- **G** and **R** combined models (so **G** and **R** → **V**):
  - Specify random intercept variance $\tau^2_{U_0}$ in **G** using RANDOM statement, then specify **R** model using REPEATED statement
  - **G** matrix = Level-2 BP variance and covariance due to U_{0i}, so **R** = Level-1 WP variance and covariance of the e_{ti} residuals
  - **R** models what's left after accounting for mean differences between persons (via the random intercept variance $\tau^2_{U_0}$ in **G**)

Syntax for Models for the Variance

- Does your model include r**andom intercept variance** $\tau^2_{U_0}$ **(for U_{0i})** ?
  - Use the **RANDOM** statement → **G matrix**
  - Random intercept models BP interindividual differences in mean Y
- What about **residual variance** σ^2_e **(for e_{ti})** ?
  - Use the **REPEATED** statement → **R matrix**
    - **WITHOUT a RANDOM statement: R is BP and WP variance together** = σ^2_T
      → Total variances and covariances (to model all variation, so **R** = **V**)
    - **WITH a RANDOM statement: R is WP variance only =** σ^2_e
      → Residual variances and covariances to model WP intraindividual variation
      → **G** and **R** put back together = **V matrix** of total variances and covariances
- The **REPEATED** statement is always there implicitly…
  - Any model **always** has at least one residual variance in **R** matrix
- But the **RANDOM** statement is only there if you write it
  - **G** matrix isn't always necessary (don't always need random intercept)

Wrapping Up: ACS Models

- Even if you just expect fluctuation over time rather than change, you still should be concerned about accurately predicting the variances and covariances across occasions

- Baseline models (from ANOVA least squares) are CS & UN:
  - Compound Symmetry: Equal variance and covariance over time
  - Unstructured: All variances & covariances estimated separately
  - CS and UN via ML or REML estimation allows missing data

- MLM gives us choices in the middle
  - Goal: Get as close to UN as parsimoniously as possible
  - **R**-only: Structure TOTAL variation in one matrix (**R** only)
  - **G**+**R**: Put constant covariance due to random intercept in **G**, then structural RESIDUAL covariance in **R** (so that **G** and **R** → **V** TOTAL)