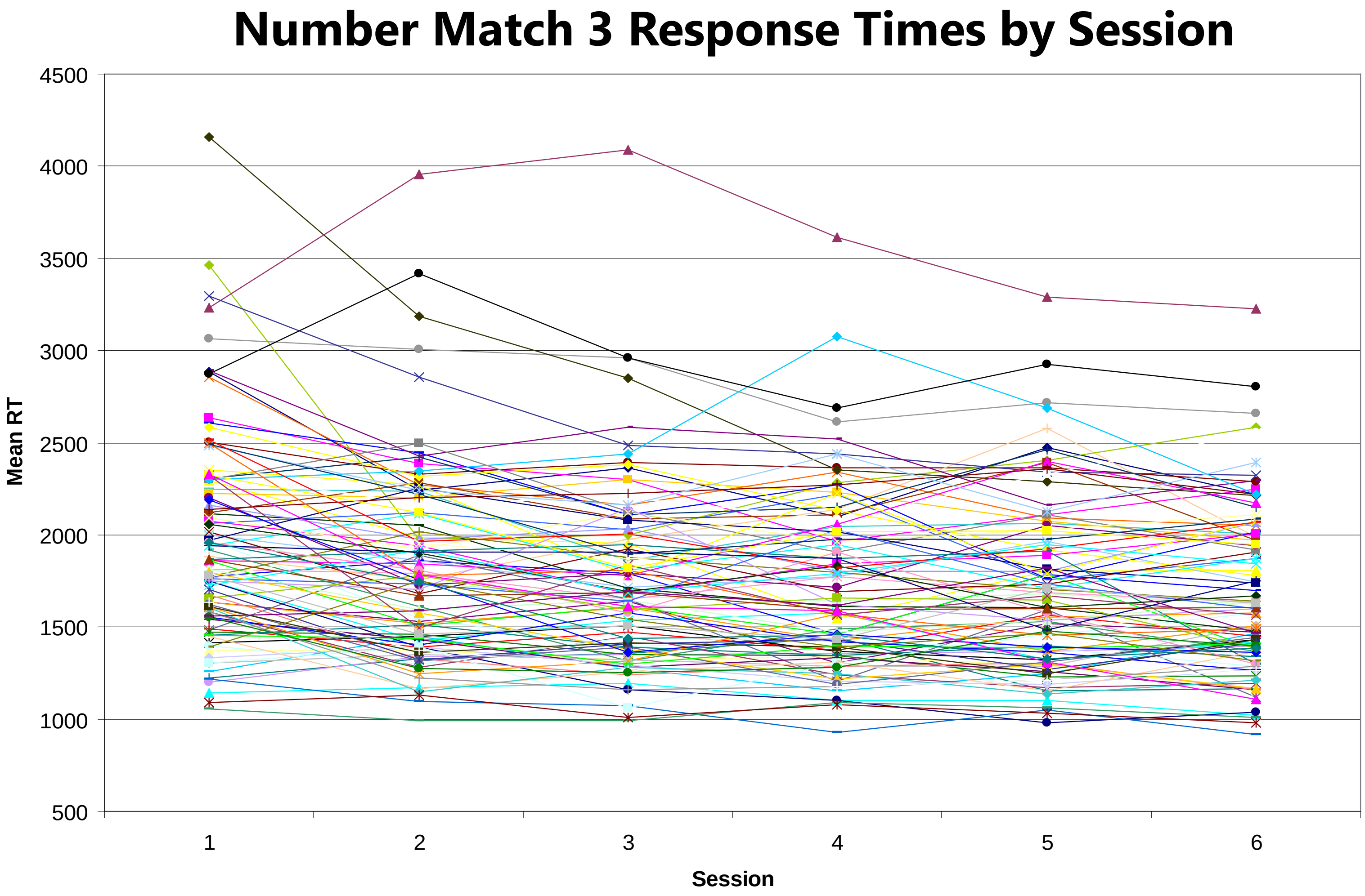
Describing Within-Person Change over Time

- Topics:
  - The Big Picture of modeling change
  - Fixed and random effects models for nonlinear change:
    - Polynomial slopes
    - Piecewise slopes
    - Nonlinear change

Example Data Individual Observed Trajectories (N = 101, *n* = 6)



The Big Picture of Longitudinal Data: Model for the Means (Fixed Effects)

- What kind of change occurs on average over "time"?
  - What is the most appropriate **metric of time**?
    - Time in study (with predictors for BP differences in time)?
    - Time since birth (age)? Time to event (time since diagnosis)?
    - Measurement occasions need not be the same across persons or equally spaced (code time as exactly as possible)
  - What kind of **theoretical process** generated the observed trajectories, and thus what kind of model do we need?
    - Linear or nonlinear? Continuous or discontinuous? Does change keep happening or does it eventually stop?
    - Many options: polynomial, piecewise, and nonlinear families

The Big Picture of Longitudinal Data: Models for the Means (Fixed Effects)

- What kind of change occurs on average over "time"? Two baseline models for comparison:
  - "**Empty**" → only a fixed intercept (predicts no change)
  - "**Saturated**" → all occasion mean differences from time 0 (ANOVA model that uses # fixed effects = *n*) *** *may not be possible in unbalanced data*

Parsimony

Empty Model:
Predicts NO change over time
1 Fixed Effect

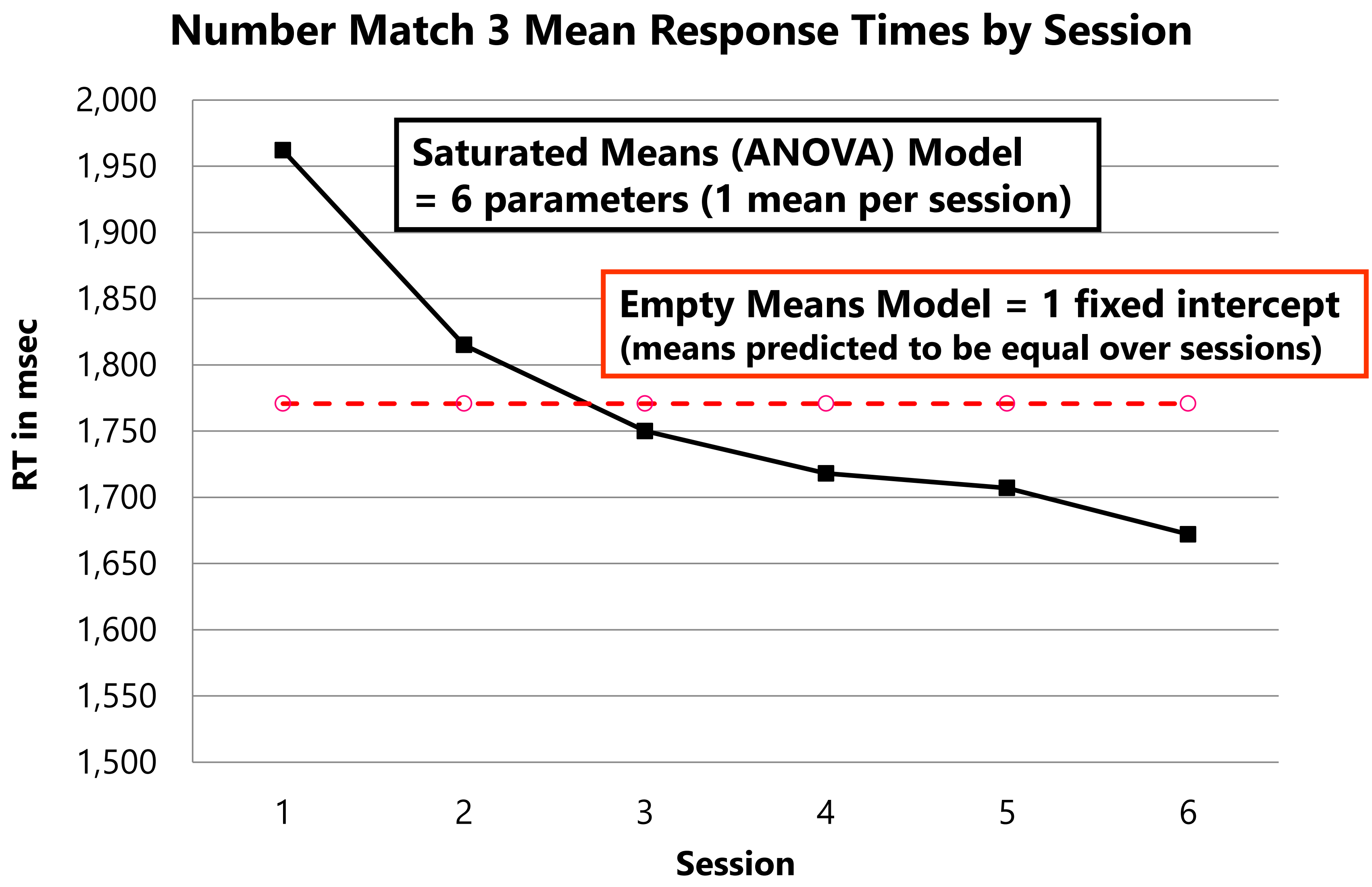
In-between options:
polynomial slopes, piecewise slopes, nonlinear models…

Saturated Means:
Reproduces mean at each occasion
 # Fixed Effects = # Occasions

Good fit

Name… that… Trajectory!

Baseline Models for the Means



The Big Picture of Longitudinal Data: Models for the Variance (Random Effects)

- *From a substantive perspective*: Are there **individual differences** in change?
  - Individual differences in the level of an outcome?
    - At what time point are individual differences in outcome level important for your hypotheses (beginning, middle, end)?
  - Individual differences in magnitude of change?
    - Each aspect of change (e.g., linear change, quadratic change) can potentially exhibit individual differences (data permitting)
- *From a statistical perspective*: What kind of pattern do the **variances and covariances** exhibit over time?
  - Do the variances increase or decrease over time?
  - Are the covariances differentially related based on time?

The Big Picture of Longitudinal Data: Models for the Variance

<u>Compound Symmetry (CS)</u>

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

Most useful model: likely somewhere in between!

<u>Unstructured (UN)</u>

$$\begin{bmatrix} \sigma^2_{T1} & \sigma_{T12} & \sigma_{T13} & \sigma_{T14} \\ \sigma_{T21} & \sigma^2_{T2} & \sigma_{T23} & \sigma_{T24} \\ \sigma_{T31} & \sigma_{T32} & \sigma^2_{T3} & \sigma_{T43} \\ \sigma_{T41} & \sigma_{T42} & \sigma_{T43} & \sigma^2_{T4} \end{bmatrix}$$

Parsimony ⟵⟶ **Good fit**

Univariate RM ANOVA

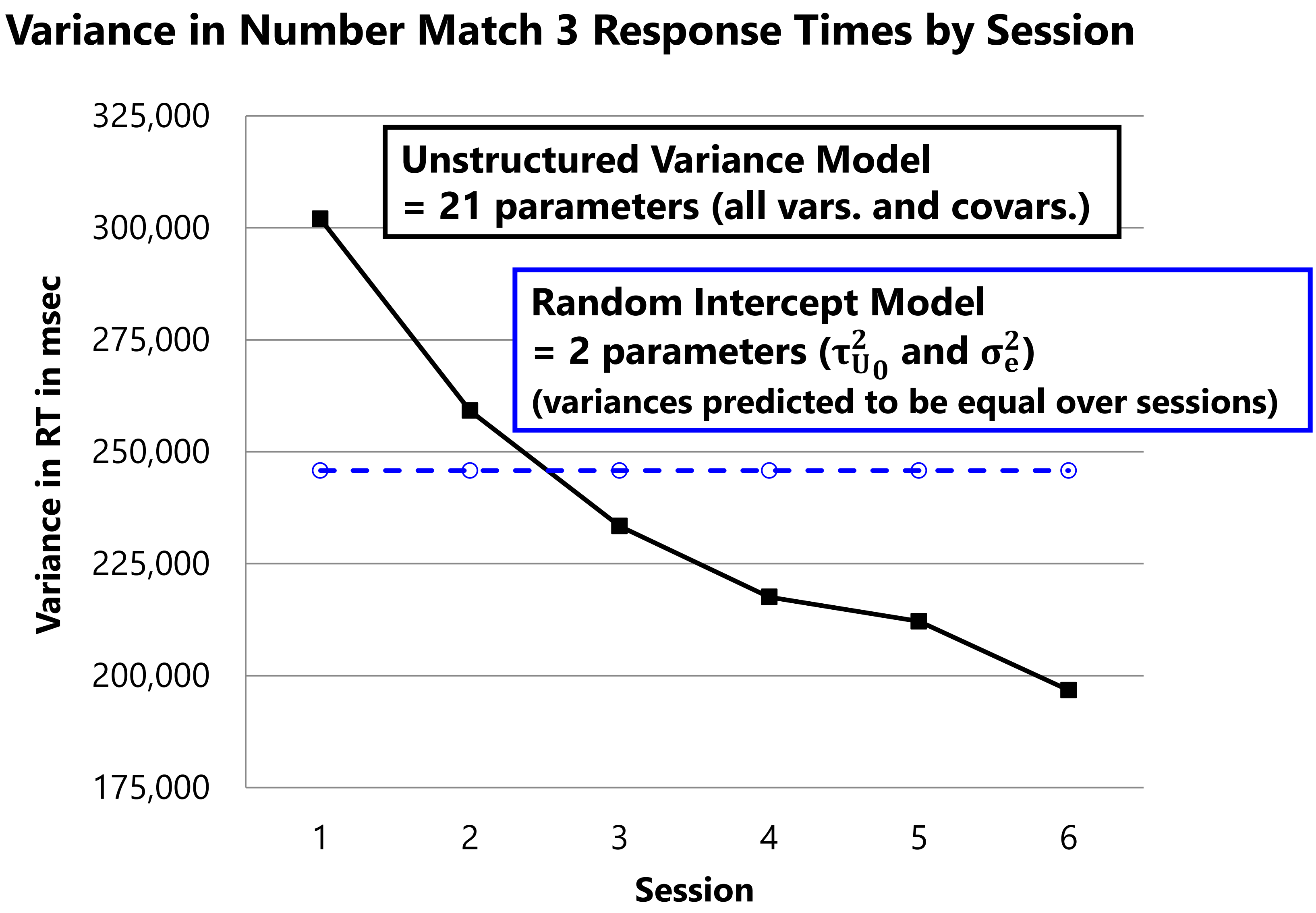
NAME …THAT … STRUCTURE!

Multivariate RM ANOVA

What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Baseline Models for the Variance



Summary: Modeling Means and Variances

- We have two tasks in describing within-person change:

- **Choose a Model for the Means**
  - What kind of change in the outcome do we have on average?
  - What kind and how many **fixed effects** do we need to predict that mean change as parsimoniously but accurately as possible?

- **Choose a Model for the Variances**
  - What pattern do the variances and covariances of the outcome show over time because of individual differences in change?
  - What kind and how many **random effects** do we need to predict that pattern as parsimoniously but accurately as possible?

New: Testing Absolute Fit in REML

- Answer key model (possible only for balanced data):
  - Means Model = Saturated Means
  - Variance Model = Unstructured R, or RI+UN($n-1$) equivalent

- Tests of absolute fit of any simpler means model against saturated means can only be done via $-2\Delta LL$ when using ML, but what if you need to use REML given small level-2 N?
  - Use a multivariate Wald test instead: add enough contrasts for occasion-specific mean differences to create saturated means, then test that group of contrasts (see example 6 for how to do so using CLASS/BY)

- Tests of absolute fit of any nested variance model against UN can be done using REML $-2\Delta LL$ if same means side (so keep the same fixed effects for time in each comparison model)

Name that trajectory… Polynomial?

- Predict **mean change** with **polynomial fixed effects of time**:
  - Linear → *constant* amount of change (up or down)
  - Quadratic → *change* in linear rate of change (acceleration/deceleration)
  - Cubic → *change* in acceleration/deceleration of linear rate of change (known in physics as jerk, surge, or jolt)
  - Terms work together to describe curved trajectories

- **Can have polynomial fixed time slopes UP TO: *n* – 1***
  - 3 occasions = 2nd order ($time^2$)= Fixed Quadratic Time or less
  - 4 occasions = 3rd order ($time^3$) = Fixed Cubic Time or less

- Interpretable polynomials past cubic are rarely seen in practice

***This rule can be broken in unbalanced data (but cautiously)

Interpreting Quadratic Fixed Effects

A Quadratic time effect is a two-way interaction: time*time

- Fixed quadratic time = "**half the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**

- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…

- The "twice" part comes from taking the derivatives of the function:

Intercept (Position) at Time T: $\hat{y}_T = 50.0 + 4.0T + 0.3T^2$

First Derivative (Velocity) at Time T: $\frac{d\hat{y}_T}{d(T)} = 4.0 + 0.6T$

Second Derivative (Acceleration) at Time T: $\frac{d^2\hat{y}_T}{d(T)} = 0.6$

Interpreting Quadratic Fixed Effects

<u>A Quadratic time effect is a two-way interaction: time*time</u>

- Fixed quadratic = "**<u>half</u> the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**
- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…
- The "twice" part also comes from what you remember about the role of interactions with respect to their constituent main effects:

$$\hat{y} = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ$$

$$\text{Effect of } X = \beta_1 + \beta_3 Z$$

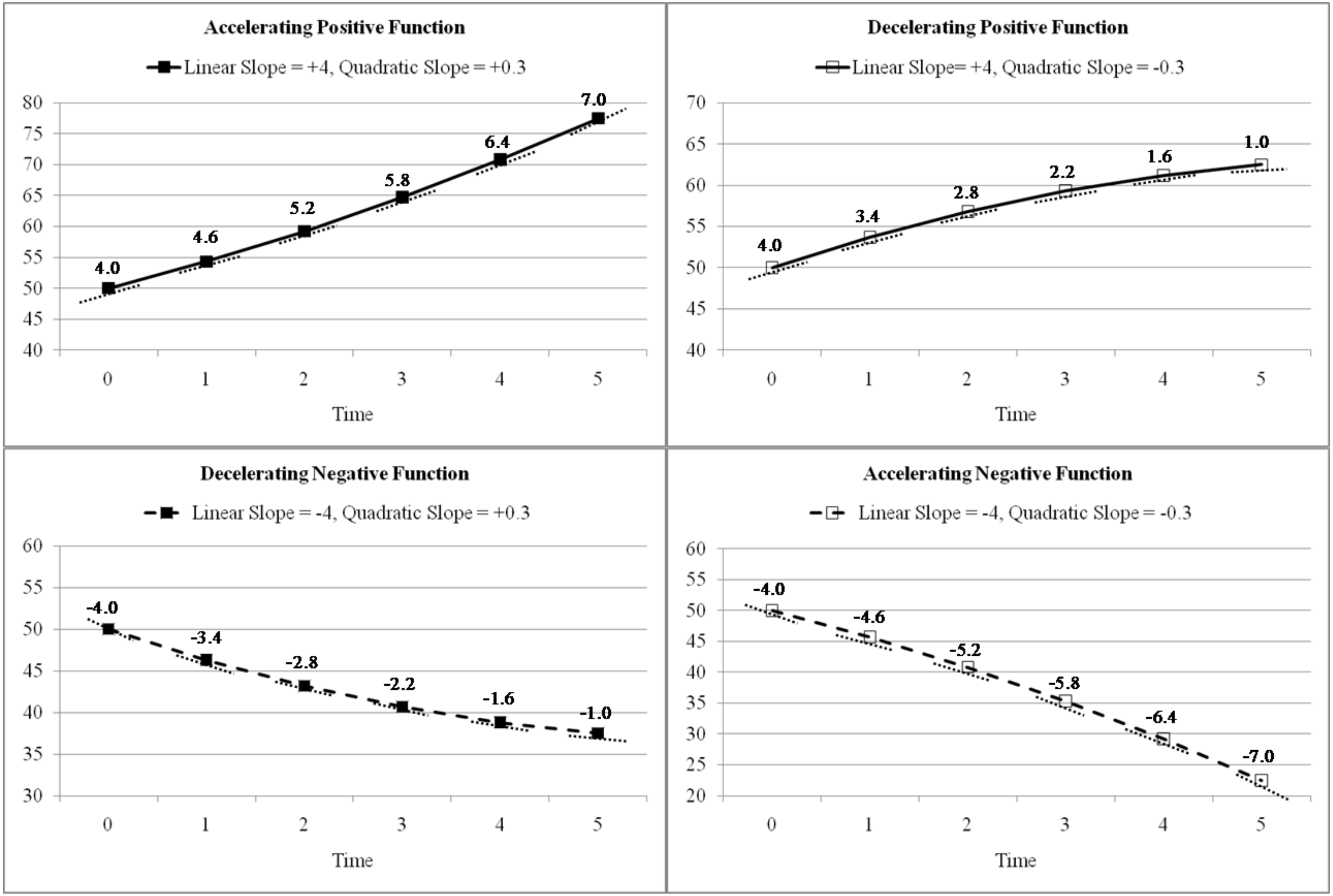
$$\text{Effect of } Z = \beta_2 + \beta_3 X$$

$$\hat{y}_T = \beta_0 + \beta_1 \text{Time}_T + \underline{\qquad} + \beta_3 \text{Time}_T^2$$

$$\text{Effect of Time}_T = \beta_1 + 2\beta_3 \text{Time}_T$$

- Because time is interacting with itself, there is no second main effect in the model for the interaction to modify as usual. So the quadratic time effect gets applied <u>twice</u> to the <u>one</u> (main) linear effect of time.

Examples of Fixed Quadratic Time Effects



Conditionality of Polynomial Fixed Time Effects

- We've seen how main effects become conditional simple effects once they are part of an interaction
- The same is true for polynomial **fixed effects of time**:
  - ***Fixed Intercept Only?***
    - Fixed Intercept = predicted mean of Y *for any occasion (= grand mean)*
  - ***Add Fixed Linear Time?***
    - Fixed Intercept = **now** predicted mean of Y from linear time *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Linear Time = mean linear rate of change *across all occasions* *(would be the same if time was centered elsewhere)*
  - ***Add Fixed Quadratic Time?***
    - Fixed Intercept = still predicted mean of Y *at time=0 (but from quadratic model)* (*would be different if time was centered elsewhere*)
    - Fixed Linear Time = **now** mean linear rate of change *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Quadratic Time = half the mean rate of acceleration or deceleration of change *across all occasions* (i.e., the linear slope changes the same over time)

Polynomial Fixed vs. Random Time Effects

- **Polynomial fixed effects** combine to describe mean trajectory over time (can have fixed slopes up to ***n* – 1**):
  - Fixed Intercept = Predicted mean level (at time 0)
  - Fixed Linear Time = Mean linear rate of change (at time 0)
  - Fixed Quadratic Time = Half of mean acceleration/deceleration in linear rate of change (2*quad is how the linear time slope changes per unit time if quadratic is highest order fixed effect)

- **Polynomial random effects** (individual deviations from the fixed effect) describe individual differences in those change parameters (can have random slopes up to ***n* – 2**):
  - Random Intercept = BP variance in level (at time 0)
  - Random Linear Time = BP variance in linear time slope (at time 0)
  - Random Quadratic Time = BP variance in half the rate of acceleration/deceleration of linear time slope (across all time if quadratic is highest-order random effect)

Random Quadratic Time Model

Level 1: $\mathbf{y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}}$

Level 2 Equations (one per β):

$\boldsymbol{\beta_{0i}} = \boldsymbol{\gamma_{00}} + \boldsymbol{U_{0i}}$

Intercept for person *i*

Fixed (mean) **Intercept**

Random (Deviation) **Intercept**

$\boldsymbol{\beta_{1i}} = \boldsymbol{\gamma_{10}} + \boldsymbol{U_{1i}}$

Linear Slope for person *i*

Fixed (mean) **Linear Slope**

Random (Deviation) **Linear Slope**

$\boldsymbol{\beta_{2i}} = \boldsymbol{\gamma_{20}} + \boldsymbol{U_{2i}}$

Quad Slope for person *i*

Fixed (mean) **Quad Slope**

Random (Deviation) **Quad Slope**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (n):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

Need $n = 4$ occasions to fit random quadratic time model

Example Sequence for Testing Fixed and Random Polynomial Effects of Time

Build up fixed and random effects simultaneously:

1. Empty Means, Random Intercept → to calculate ICC
2. Fixed Linear, Random Intercept → check fixed linear *p*-value
3. Random Linear → check $-2\Delta LL(df \approx 2)$ for random linear variance
4. Fixed Quadratic, Random Linear → check fixed quadratic *p*-value
5. Random Quadratic → check $-2\Delta LL(df \approx 3)$ for random quadratic variance
6. …….

*** In general: Can use **REML** for all models, so long as you:

→ Test significance of new **fixed** effects by their ***p*-values**

→ Test significance of new **random** effects in separate step by **−2ΔLL**

→ Also see if AIC and BIC are smaller when adding random effects

Conditionality of Polynomial Random Effects

- We saw previously that lower-order fixed effects of time are conditional on higher-order polynomial fixed effects of time
- The same is true for polynomial **random effects of time**:
  - ***Random Intercept Only?***
    - Random Intercept = BP variance *for any occasion* in predicted mean Y (= variance in grand mean because individual lines are parallel)
  - ***Add Random Linear Time?***
    - Random Intercept = **now** BP variance *at time=0* in predicted mean Y *(would be different if time was centered elsewhere)*
    - Random Linear Time = BP variance *across all occasions* in linear rate of change *(would be the same if time was centered elsewhere)*
  - ***Add Random Quadratic Time?***
    - Random Intercept = still BP variance *at time=0* in predicted mean Y
    - Random Linear Time = **now** BP variance *at time=0* in linear rate of change *(would be different if time was centered elsewhere)*
    - Random Quadratic Time = BP variance *across all occasions* in half of accel/decel of change *(would be the same if time was centered elsewhere)*

Random Effects Allowed by #Occasions

	Data		G Matrix	R Matrix	Variance Model # Parameters
n=2 occasions **3** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} \tau_{U_0}^2 \end{bmatrix}$ **Random Intercept only**	$\begin{bmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}$	**2**
n=3 occasions **6** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & & \\ \sigma_{21} & \sigma_2^2 & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} \tau_{U_0}^2 & \\ \tau_{U_{01}} & \tau_{U_1}^2 \end{bmatrix}$ **Up to 1 Random slope**	$\begin{bmatrix} \sigma_e^2 & 0 & 0 \\ 0 & \sigma_e^2 & 0 \\ 0 & 0 & \sigma_e^2 \end{bmatrix}$	**4**
n=4 occasions **10** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & & & \\ \sigma_{21} & \sigma_2^2 & & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{bmatrix}$	$\longrightarrow$	$\begin{bmatrix} \tau_{U_0}^2 & & \\ \tau_{U_{01}} & \tau_{U_1}^2 & \\ \tau_{U_{02}} & \tau_{U_{12}} & \tau_{U_2}^2 \end{bmatrix}$ **Up to 2 Random slopes**	$\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$	**7**

Predicted **V** Matrix from Polynomial Random Effects Models

- **<u>Random linear model?</u>** Variance has a **quadratic** dependence on time
  - Variance will be at a minimum when time = $-\text{Cov}(U_0,U_1)/\text{Var}(U_1)$, and will increase parabolically and symmetrically over time
  - **Predicted variance** at each occasion and covariance between A and B:

$$\text{Var}(y_{time}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0,U_1)(\mathbf{time_t}) + \text{Var}(U_1)(\mathbf{time_t^2})$$

$$\text{Cov}(y_A,y_B) = \text{Var}(U_0) + \text{Cov}(U_0,U_1)(A + B) + \text{Var}(U_1)(AB)$$

- **<u>Random quadratic model?</u>** Variance has a **quartic** dependence on time

$$\text{Var}(y_{time}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0,U_1)(\mathbf{time_t}) + \text{Var}(U_1)(\mathbf{time_t^2}) + 2\text{Cov}(U_0,U_2)(\mathbf{time_t^2}) + 2\text{Cov}(U_1,U_2)(\mathbf{time_t^3}) + \text{Var}(U_2)(\mathbf{time_t^4})$$

$$\text{Cov}(y_A,y_B) = \text{Var}(U_0) + \text{Cov}(U_0,U_1)(A + B) + \text{Var}(U_1)(AB) + \text{Cov}(U_0,U_2)(A^2 + B^2) + \text{Cov}(U_1,U_2)[(AB^2)+(A^2B)] + \text{Var}(U_2)(A^2B^2)$$

- *The point of the story: random effects of time are a way of allowing the variances and covariances to differ over time in specific, time-dependent patterns (that result from differential individual change over time).*

Rules for Polynomial Models (and in general for fixed and random effects)

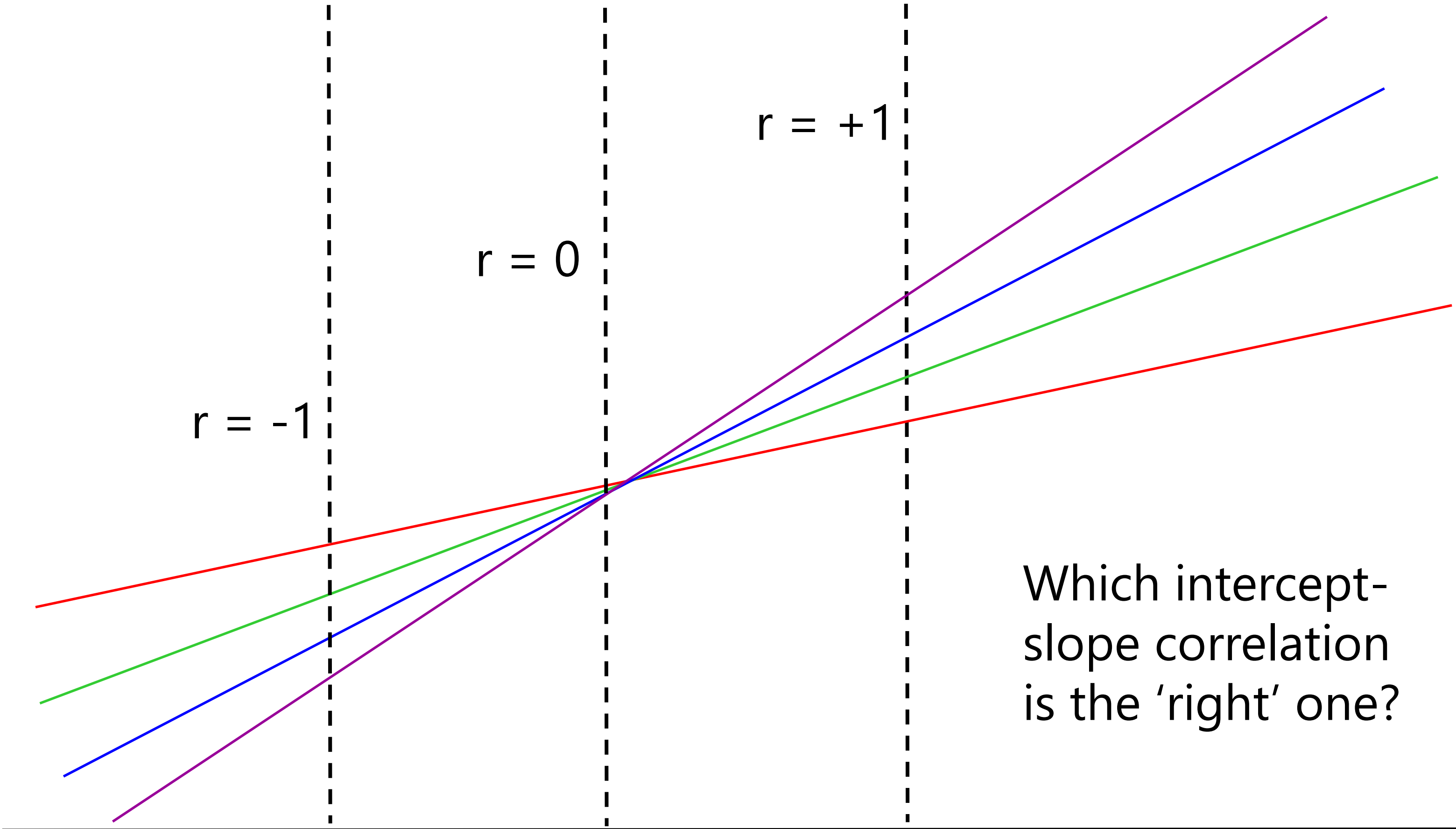
- On the same side of the model (means or variances side), lower-order effects stay in EVEN IF NONSIGNIFICANT (for correct interpretation)
  - e.g., Significant *fixed* quadratic? Keep the *fixed* linear
  - e.g., Significant *random* quadratic? Keep the *random* linear

- Also remember—you can have a significant random effect EVEN IF the corresponding fixed effect is not significant (keep it anyway):
  - e.g., Fixed linear not significant, but random linear is significant?
→ No linear change *on average*, but significant individual differences in change

- Language: A random effect supersedes a fixed effect:
  - If <u>Fixed</u> = intercept, linear, quad; <u>Random</u> = intercept, linear, quad?
    - Call it a "Random quadratic model" (implies everything beneath those terms)
  - If <u>Fixed</u> = intercept, linear, quad; <u>Random</u> = intercept, linear?
    - Call it a "Fixed quadratic, random linear model" (distinguishes no random quad)

- Intercept-slope correlation depends largely on centering of time…

Correlation between Random Intercept and Random Linear Slope depends on time 0



Which intercept-slope correlation is the 'right' one?

!! Nonparallel lines will eventually cross.

Correlations among polynomial slopes

Session Centered at 1:

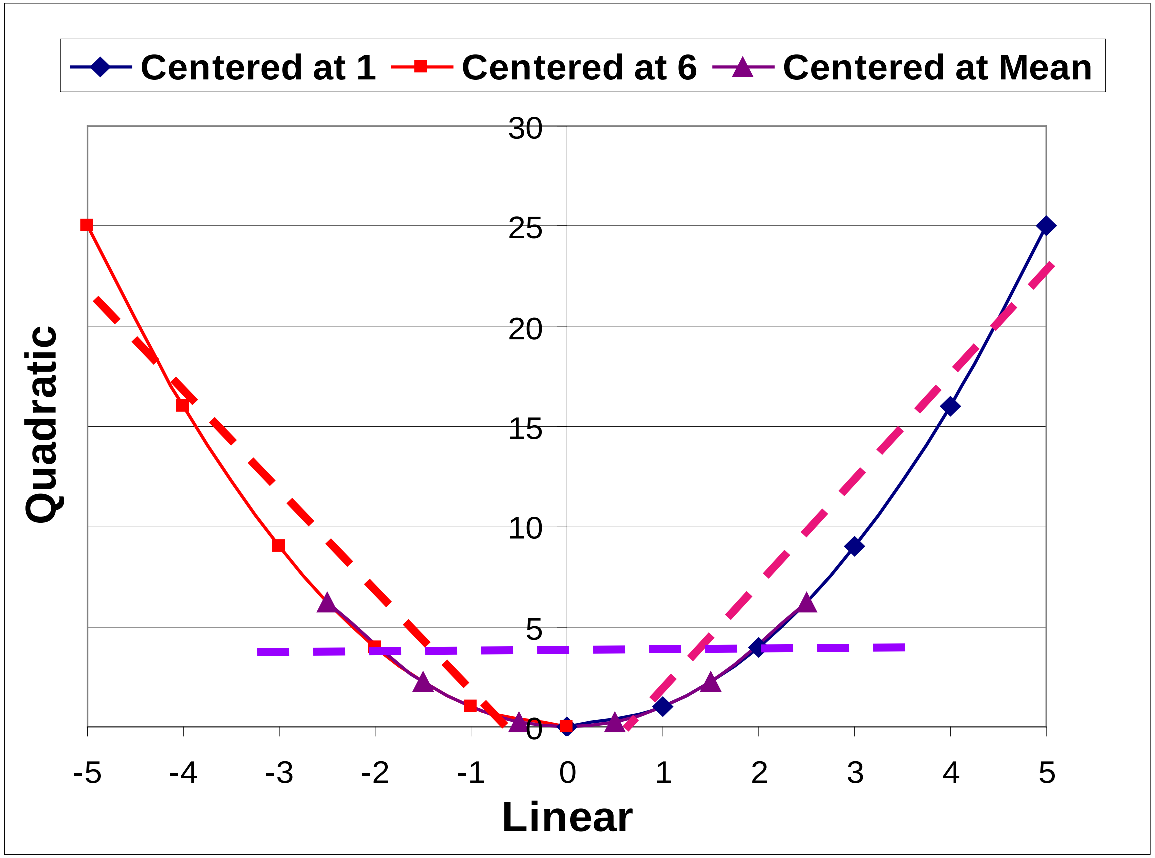
Session	Linear	Quadratic
1	0	0
2	1	1
3	2	4
4	3	9
5	4	16
6	5	25

Session Centered at 6:

Session	Linear	Quadratic
1	-5	25
2	-4	16
3	-3	9
4	-2	4
5	-1	1
6	0	0

Session Centered at Mean:

Session	Linear	Quadratic
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	2.25
6	2.5	6.25



Correlations among polynomial effects of time can be induced by centering time near the start or near the end.

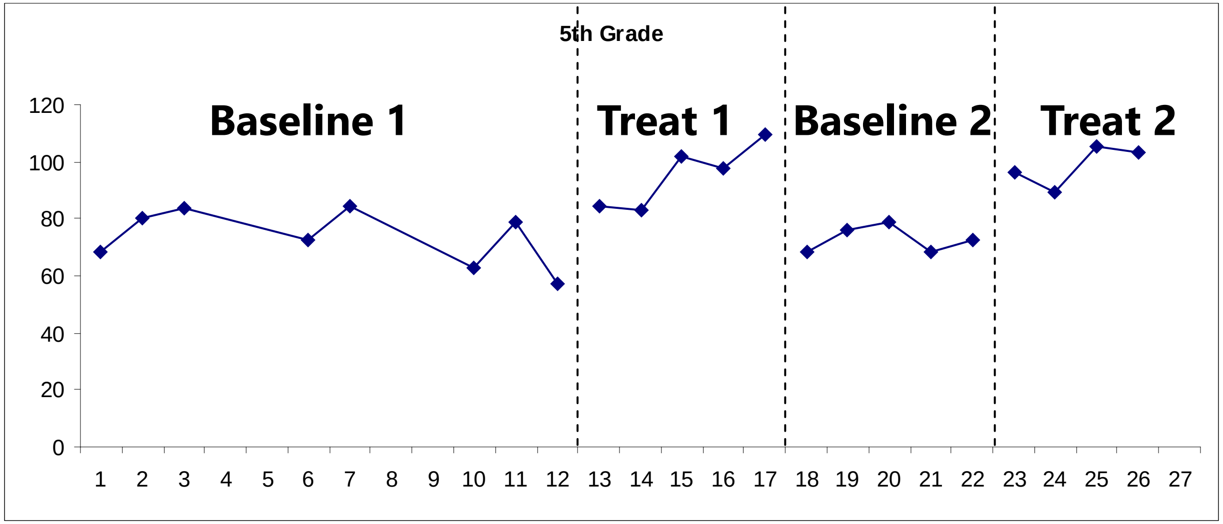
Therefore, these correlations will be *most* interpretable when centering time at its mean instead.

Summarizing so far…

- Modeling within-person change involves specifying effects of time for both sides of the model
  - **Fixed effects in model for the means:**
    - What kind of change am I observing on average?
    - What kind of trajectory will reproduce those means?
  - **Random effects (and residuals) in model for the variances:**
    - What kind of individual differences in change am I observing?
    - How many random effects do I need to reproduce the observed pattern of variances and covariances over time?
- One option: Polynomial models (linear, quadratic, cubic)
  - Terms work together to describe non-linear trajectories
  - Careful with the covariances among random effects, though
- Coming next: Piecewise slopes and truly nonlinear change…

Other Random Effects Models of Change

- **Piecewise models**: Discrete slopes for discrete phases of time
  - Separate terms describe sections of overall trajectories
  - Useful for examining change in intercepts and slopes before/after discrete events (changes in policy, interventions)
  - **Must know where the break point is ahead of time!**



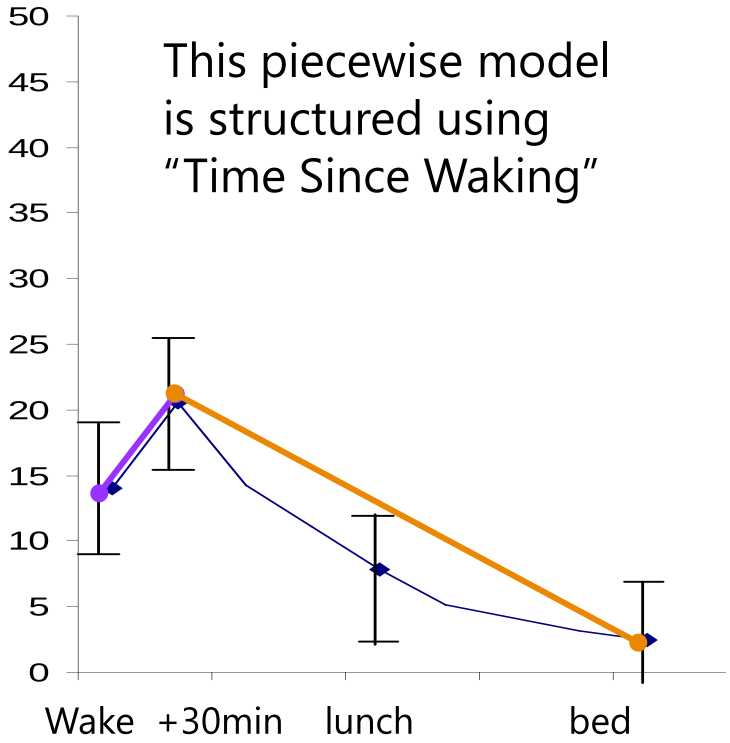
Piecewise Model:

4 slopes
(one per phase)

3 "jumps"
(shift in intercept between phases)

Example of Daily Cortisol Fluctuation: Morning Rise and Afternoon Decline

Average Trajectories



SAS Code to create two piecewise slopes from continuous time of day in stacked data:

```
IF occasion=1 THEN DO;
    P1=0;                P2=0; END;
IF occasion=2 THEN DO;
    P1= time2-time1; P2=0; END;
IF occasion=3 THEN DO;
    P1= time2-time1; P2=time3-time2; END;
IF occasion=4 THEN DO;
    P1= time2-time1; P2=time4-time2; END;
```

Note that a quadratic slope may be necessary for the afternoon decline slope!

Random Two-Slope Piecewise Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Slope1_{ti} + \beta_{2i}Slope2_{ti} + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

Intercept for person *i* — Fixed (mean) **Intercept** — Random (Deviation) **Intercept**

$\beta_{1i} = \gamma_{10} + U_{1i}$

Slope1 for person *i* — Fixed (mean) **Slope1** — Random (Deviation) **Slope1**

$\beta_{2i} = \gamma_{20} + U_{2i}$

Slope2 for person *i* — Fixed (mean) **Slope2** — Random (Deviation) **Slope2**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

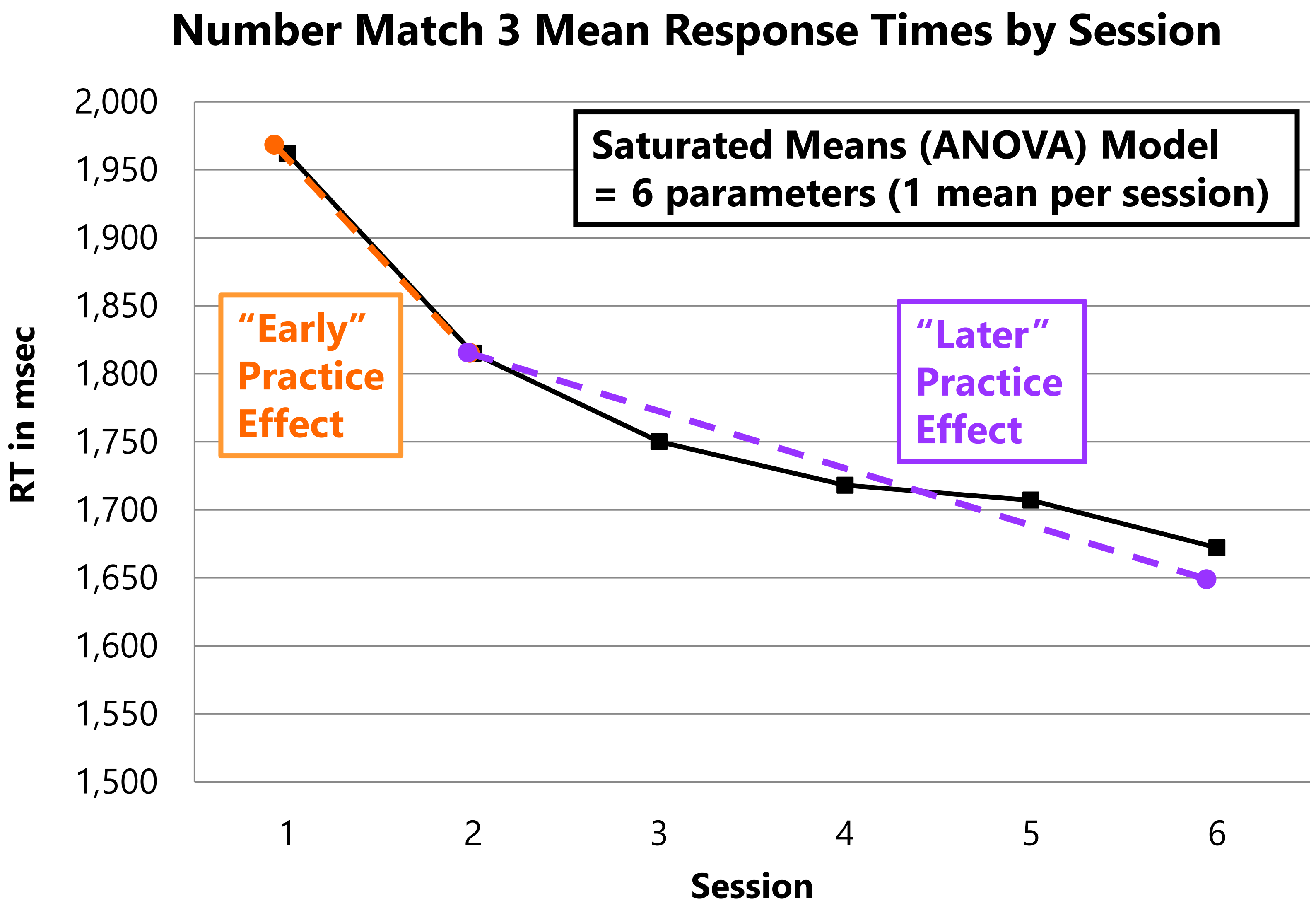
Number of Possible Slopes by Number of Occasions (*n*):

\# Fixed slopes = $n - 1$

\# Random slopes = $n - 2$

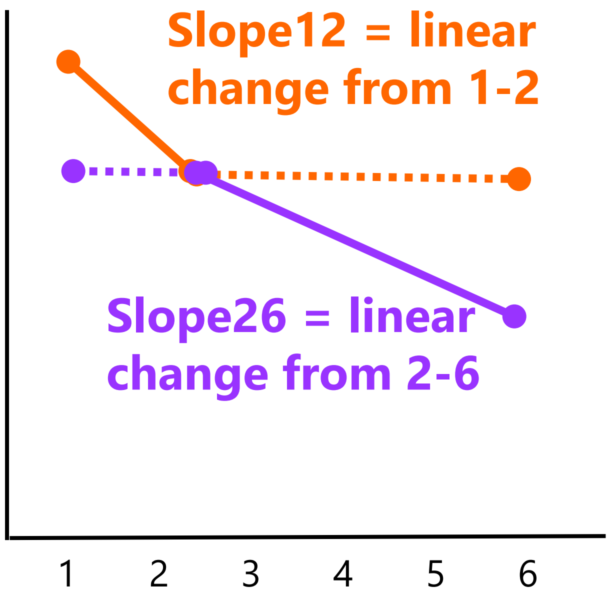
Need $n = 4$ occasions to fit random two-slope model

What kind of piecewise model could predict our example data mean change across sessions?



Piecewise Models: Two Direct Slopes

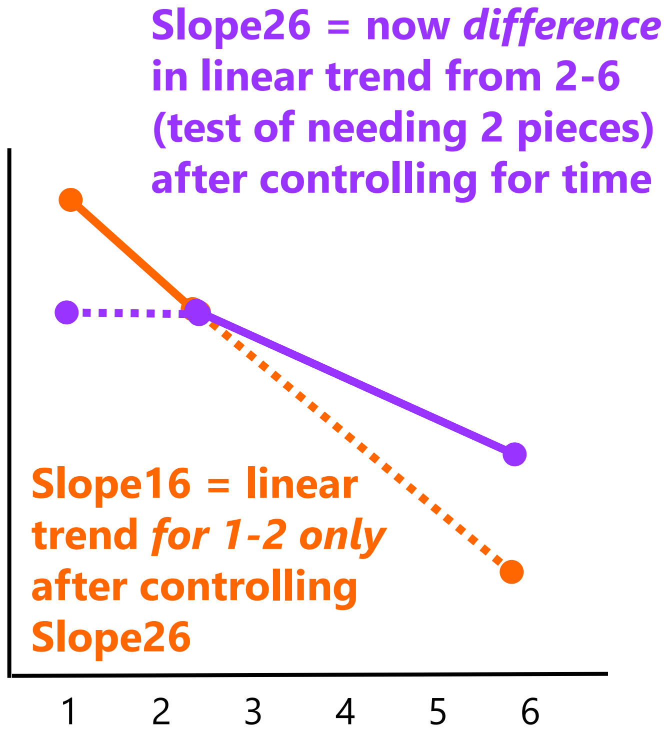
- "Early Practice Slope" and "Later Practice Slope"
- Use to specify slopes through each discrete phase directly
- Session (1-6) gets recoded into 2 new time predictor variables, as shown below:



Session	1	2	3	4	5	6
Early Practice → Slope12 =	0	1	1	1	1	1
Later Practice → Slope26 =	0	0	1	2	3	4

Piecewise Models: Slope +Deviation Slope

- "Linear Time Slope" and "Deviation Slope"
- Use to test if multiple slopes are needed
- Initial slope predictor is coded differently, second slope predictor is same:



Session		**1**	**2**	**3**	**4**	**5**	**6**
Time	**→ Slope16 =**	**0**	**1**	**2**	**3**	**4**	**5**
Deviation	**→ Slope26 =**	**0**	**0**	**1**	**2**	**3**	**4**

2 Direct Slopes Model: Random Effects

- Parameters directly **represent each part** of trajectory:

- **Fixed effects** for mean change over time (3):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope12 = expected linear rate of change from 1 to 2
  - Fixed Slope26 = expected linear rate of change from 2 to 6

- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level
    when both slopes = 0 (at Session 1)
  - Random Slope12 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in linear slope from 2 to 6

Slope + Deviation Slope: Random Effects

- Parameters directly **differences across parts** of trajectory:

- **Fixed effects** for mean change over time (3):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope16 = expected linear rate of change from 1 to 2 (after controlling for slope26)
  - Fixed Slope26 = expected **extra** linear rate of change from 2 to 6 (after controlling for slope16, which is just time)

- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level when both slopes = 0 (at Session 1)
  - Random Slope16 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in **extra** linear slope from 2 to 6

Saturated Means via Piecewise Slopes Models

- You can fit **fixed** piecewise slopes up to *n*–1, but only **random** piecewise slopes up to *n*–2:
  - 3 occasions? up to 2 fixed pieces, but only 1 random piece
  - 4 occasions? up to 3 fixed pieces, but only 2 random pieces
  - *n*–1 fixed pieces will perfectly reproduce observed means

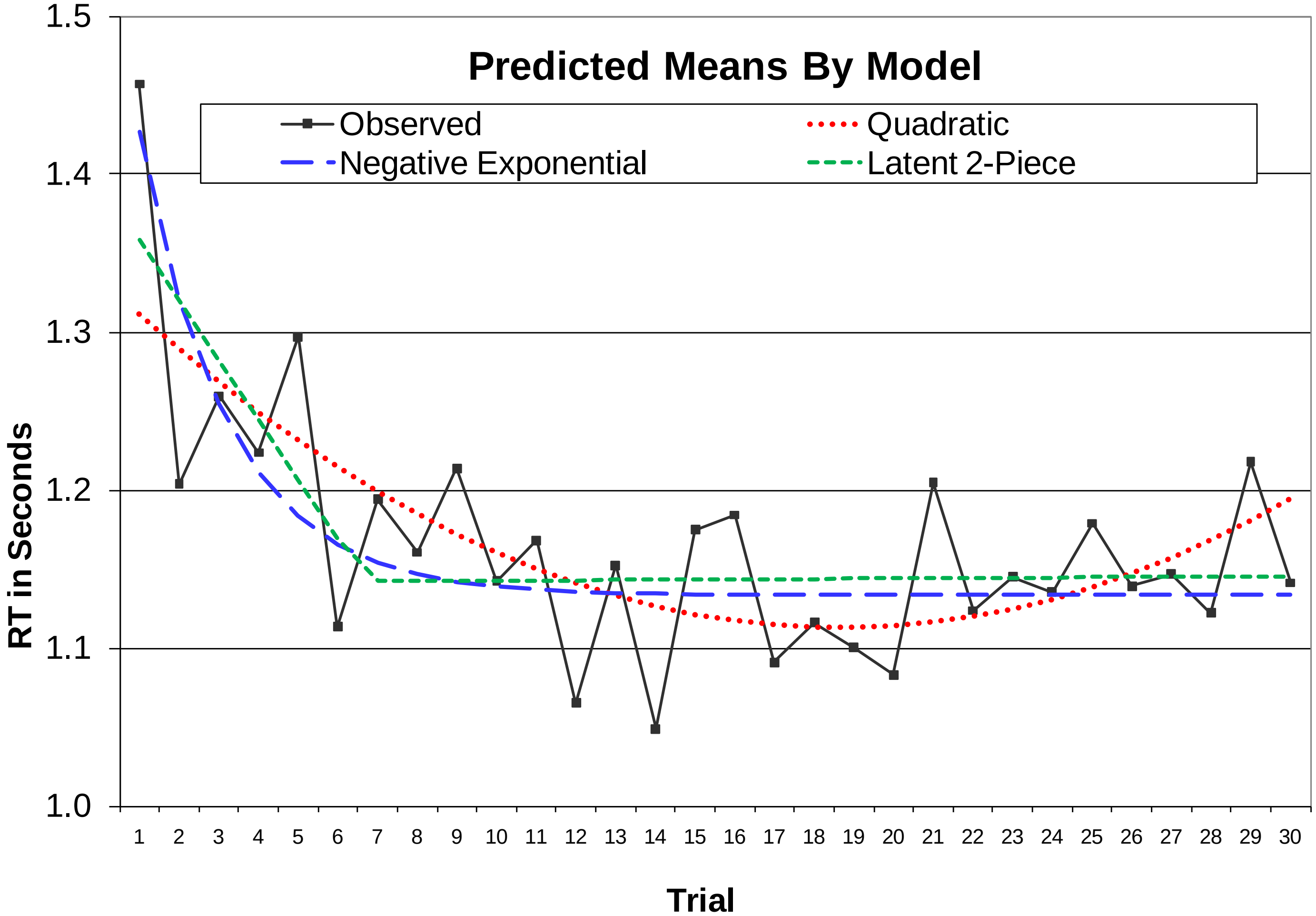
- Given this constraint (and balanced data), you should consider some of the ACS models as well:
  - Example: *n*=3 → Model for the means = 2 fixed pieces, Model for the Variances could be….
    - UN, CSH, CS (Random Intercept Only), Random Intercept + Random Slope12, OR Random Intercept + Random Slope23
    - Everything is nested within UN; can also use AIC and BIC to choose

Summary: Piecewise Slopes Models

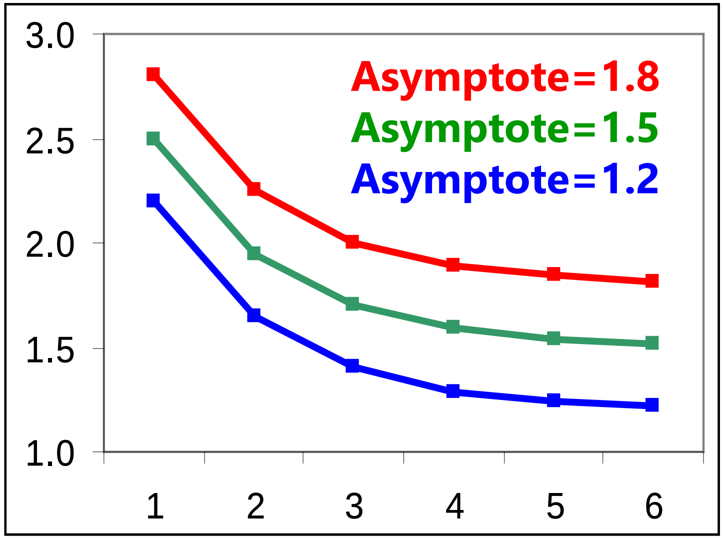
- Piecewise models are useful for discontinuous trajectories (empirically or based on the study design)
  - Use slope + deviation slope(s) to test if > 1 slope is necessary
- If all effects are random, the slope + deviation slope and the direct slopes versions of the models will be equivalent
  - Select the one that has the random effects variance you want to predict
- Keep all the pieces in the model (even if non-significant) in order to maintain a correct interpretation of each
- Each piece can be linear or non-linear as needed
  - e.g., piece1 + piece2 + piece2^2 → linear, then non-linear trajectory
- You may also need to test for a 'drop' or 'jump' in intercept at the break point in addition to change in slope, data permitting
  - Planning on doing piecewise models? They can be tricky…
PLEASE let me help you set up the predictors to do so!

Other Random Effects for Change

- **Truly nonlinear models**: Non-additive terms to describe change
  - Models can include **asymptotes** (so change can "shut off" as needed)
  - Include **power** and **exponential** functions (see chapter 6 for references)



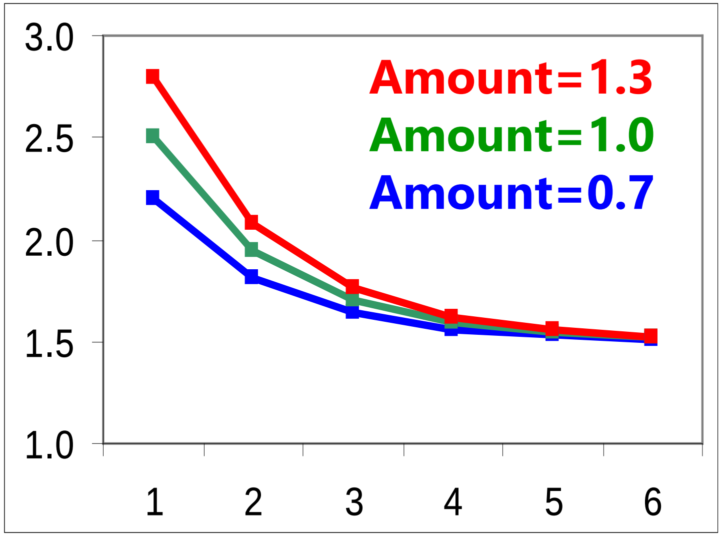
(Negative) Exponential Model Parameters

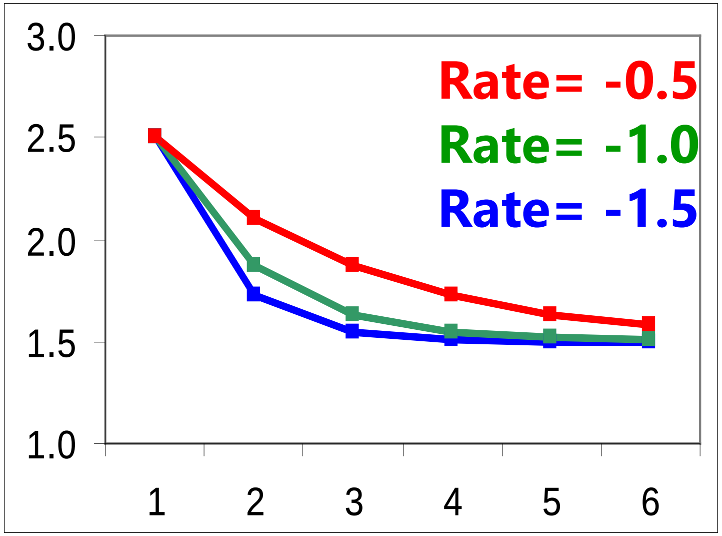
1)


1) Different **Asymptotes**, same amount and rate

2) Different **Amounts**, same asymptote and rate

3) Different **Rates**, same asymptote and amount

2)


3)


Exponential Models

- The name positive or negative reflects whether the data are moving away or towards asymptote
  - *Accelerating* trajectory (up or down) = "positive" exponential
  - *Decelerating* trajectory (up or down) = "negative" exponential
- Amount reflects distance from asymptote to time 0, multiplied by exp(rate*time)
  - Decrease across time to asymptote = positive amount
  - Increase across time to asymptote = negative amount
- Amount can also be replaced by an intercept
  - Asymptote + Amount = Intercept
- Cannot be estimated in SAS PROC MIXED given its nonlinear parameters (use SAS PROC NLMIXED instead)

Exponential Model (3 Random Effects)

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}*\exp(\beta_{2i}*Time_{ti}) + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

Asymptote for person *i*

Fixed (mean) **Asymptote**

Random (Deviation) **Asymptote**

$\beta_{1i} = \gamma_{10} + U_{1i}$

Amount for person *i*

Fixed (mean) **Amount**

Random (Deviation) **Amount**

$\beta_{2i} = \gamma_{20} + U_{2i}$

Rate for person *i*

Fixed (mean) **Rate**

Random (Deviation) **Rate**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (*n*):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

Also need 4 occasions to fit random exponential model

(Likely need way more occasions to find U_{2i}, though)

Nonlinear Models

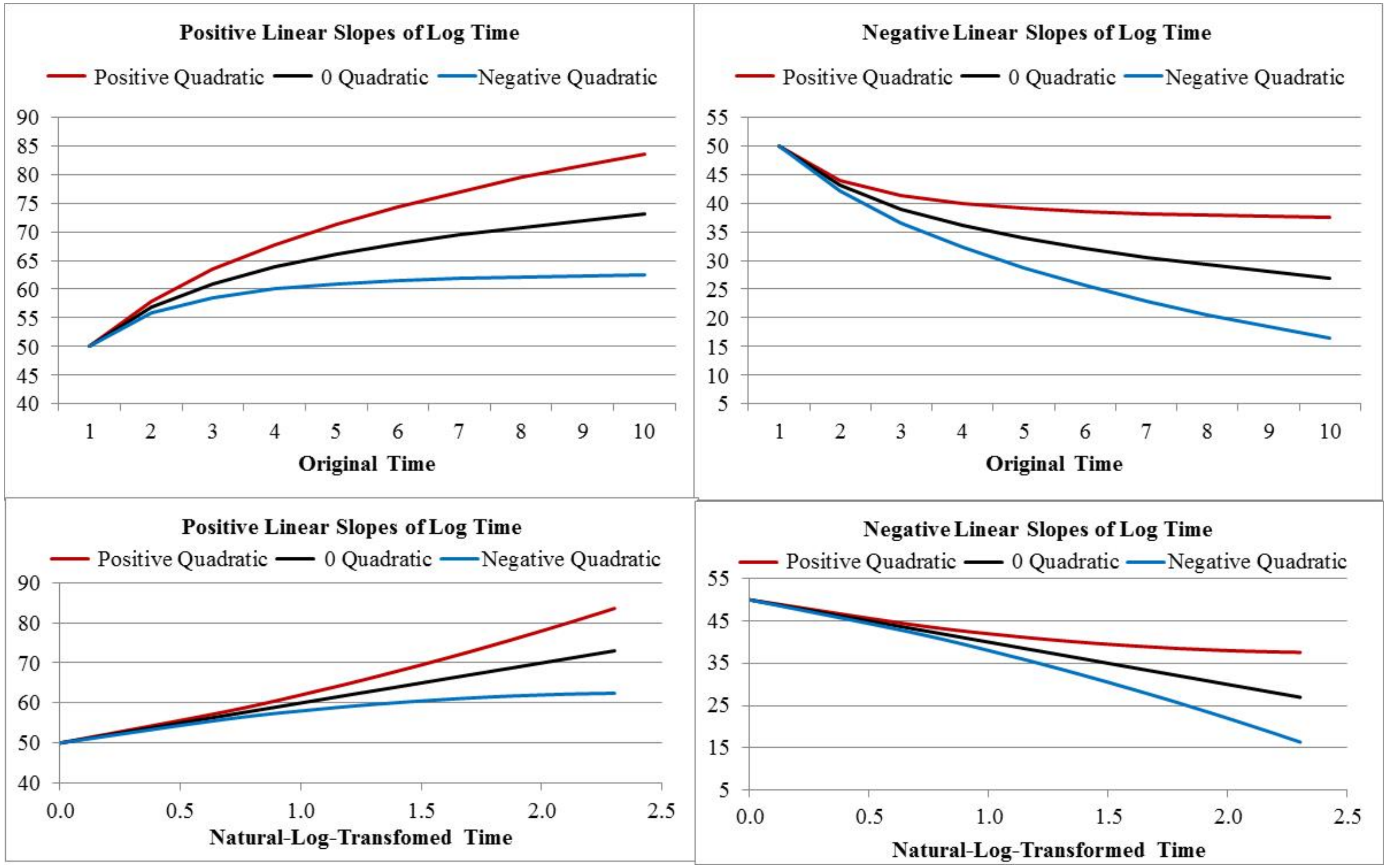
- Not all forms of change fit polynomial models
  - What goes up must come back down (and vice-versa)
  - Sometimes change needs to “shut off” (need asymptotes)

- Many kinds of truly nonlinear models can be used for longitudinal data
  - Linear in variables vs. linear in parameters (exp → nonlinear)
  - Logistic, power, exponential… see end of chapter 6 for ideas

- Require extra steps to evaluate estimation quality
  - Start values are needed, especially for random variances
  - Check that “gradient” values are as close to 0 as possible (partial first derivative of that parameter in LL function)

How to Mimic an Exponential Model

If you need to use REML, a predictor of natural-log-transformed time may be a good substitute for a truly nonlinear model



A <u>linear</u> effect of log time (black lines) predicts an <u>exponential</u> curve across *original* time.

Quadratic effects of log time (red or blue lines) can speed up or slow down the curve.

Bottom: There is a linear relationship between log-time and the outcome.

Which change family should I choose?

- Within a given family, nested models can usually be compared to judge the need for each parameter
  - e.g., linear vs. quadratic? One slope vs. two slopes?
  - Usual nested model comparison rules apply (*p*-values for fixed effects, −2ΔLL tests for random effects)
  - When using REML, you can test absolute fit of each side separately if you have balanced data to see if you are "there yet"

- Between families, however, alternative models of change may not be nested, so deciding among them can be tricky
  - e.g., quadratic vs. two-slope vs. log time vs. exponential?
  - Use ML AIC and BIC to see what is "preferred" across the families
  - In balanced data, you can also compare each alternative to a saturated means, UN model using ML as test of absolute fit
  - Also consider plausibility of alternative models in terms of both data predictions and theoretical predictions in deciding