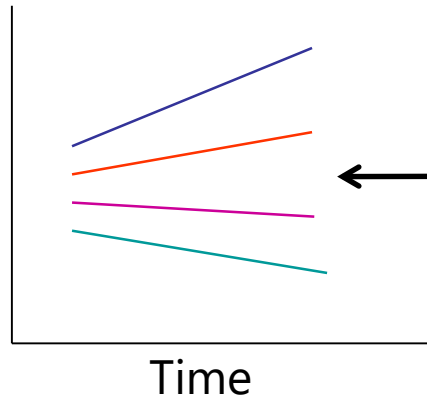


# Describing Within-Person Fluctuation over Time using Alternative Covariance Structures

- Today's Class:
  - The Big Picture
  - ACS models using the **R** matrix only
  - Introducing the **G**, **Z**, and **V** matrices
  - ACS models combining the **G** and **R** matrices

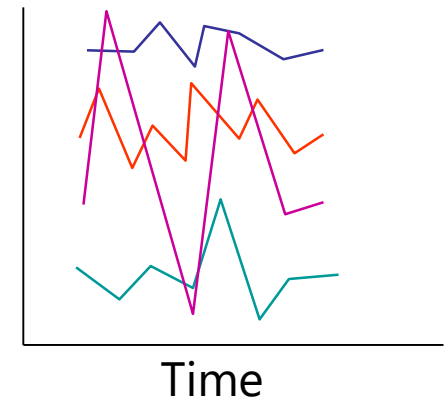
# Modeling Change vs. Fluctuation

Pure WP Change



Our focus  
right now

Pure WP Fluctuation



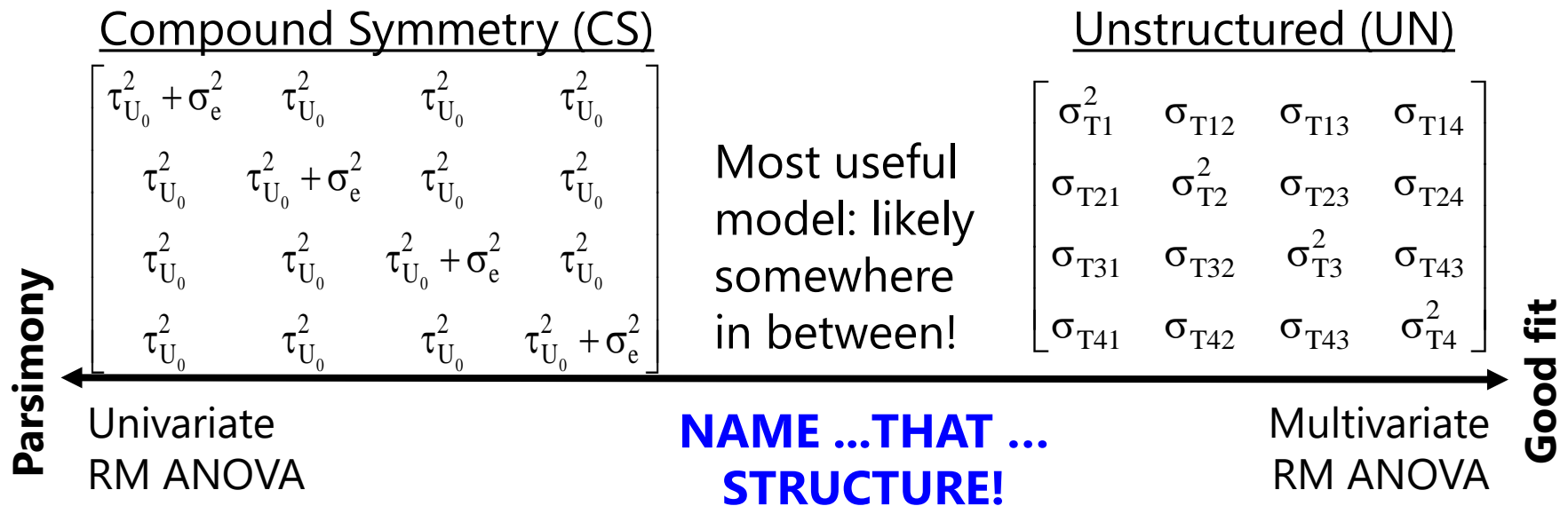
## Model for the Means:

- WP Change → describe pattern of *average* change (over "time")
- **WP Fluctuation** → \*may\* not need anything (if no systematic change)

## Model for the Variance:

- WP Change → describe *individual differences* in change (random effects)  
→ this allows variances and covariances to differ over time
- **WP Fluctuation** → describe pattern of variances and covariances over time

# Big Picture Framework: Models for the Variance in Longitudinal Data



***What is the pattern of variance and covariance over time?***

CS and UN are just two of the many, many options available within MLM, including *random effects models* (for change) and ***alternative covariance structure models*** (for fluctuation).

# Relative Model Fit by Model Side

- Nested models (i.e., in which one is a subset of the other) can now differ from each other in two important ways
- **Model for the Means** → which predictors and which fixed effects of them are included in the model
  - Does not require assessment of relative model fit using LL or  $-2LL$  (can still use univariate or multivariate Wald tests for this)
- **Model for the Variance** → what the pattern of variance and covariance of residuals from the same unit should be
  - DOES require assessment of relative model fit using LL or  $-2LL$
  - Cannot use the Wald test  $p$ -values that show up on the output for testing significance of variances because those  $p$ -values are use a two-sided sampling distribution for what the variance could be (but variances cannot be negative, so those  $p$ -values are not valid)

# Comparing Models for the Variance

- ACS models require assessment of **relative model fit**: how well does the model fit relative to other possible models?
- Relative fit is indexed by overall model **log-likelihood (LL)**:
  - Log of likelihood for each person's outcomes given model parameters
  - Sum log-likelihoods across all independent persons = **model LL**
  - Two flavors: Maximum Likelihood (ML) or Restricted ML (REML)
- What you get for this on your output varies by software...
- Given as  $-2 \times \log$  likelihood ( $-2LL$ ) in SAS or SPSS MIXED:  
 $-2LL$  gives BADNESS of fit, so **smaller** value = better model
- Given as just log-likelihood (LL) in STATA MIXED and Mplus:  
**LL** gives GOODNESS of fit, so **bigger** value = better model

# Comparing Models for the Variance

- **Two main questions in choosing a model for the variance:**
  - How does the variance of the residuals differ across occasions?
  - How are the residuals from the same sampling unit correlated?
- Nested models are compared using a **“likelihood ratio test”**:  
**–2ΔLL test** (aka, “ $\chi^2$  test” in SEM; “deviance difference test” in MLM)

“fewer” = from model with fewer parameters  
“more” = from model with more parameters

Results of 1. & 2. must  
be positive values!

1. Calculate **–2ΔLL**: if given  $-2LL$ , do  $-2\Delta LL = (-2LL_{\text{fewer}}) - (-2LL_{\text{more}})$   
if given  $LL$ , do  $-2\Delta LL = -2 * (LL_{\text{fewer}} - LL_{\text{more}})$
2. Calculate **Δdf** = (# Params<sub>more</sub>) – (# Params<sub>fewer</sub>)
3. **Compare –2ΔLL to  $\chi^2$  distribution with df = Δdf**
4. Get  $p$ -value from CHIDIST in excel or LRTEST option in STATA

# Comparing Models for the Variance

- What your  $p$ -value for the  $-2\Delta LL$  test means:
  - If you **ADD** parameters, then your model can get **better** (if  $-2\Delta LL$  test is significant) or **not better** (not significant)
  - If you **REMOVE** parameters, then your model can get **worse** (if  $-2\Delta LL$  test is significant) or **not worse** (not significant)
- Nested or non-nested models can also be compared by **Information Criteria** that also reflect model parsimony
  - No significance tests or critical values, just “smaller is better”
  - **AIC** = Akaike IC =  $-2LL + 2 * (\#parameters)$
  - **BIC** = Bayesian IC =  $-2LL + \log(N) * (\#parameters)$
  - What “parameters” means depends on flavor (except in STATA):
    - ML = ALL parameters; REML = variance model parameters only

$N = \#$ level-2 units
---------------------------

# Alternative Covariance Structure Models

- Useful in predicting patterns of variance and covariance that arise from fluctuation in the outcome across time:
  - **Variiances:** Same (homogeneous) or different (heterogeneous)?
  - **Covariances:** Same or different? If different, what is the pattern?
    - Models with heterogeneous variances predict correlation instead of covariance because covariances will differ when variances differ
  - Often don't need any fixed effects for systematic effects of time in the model for the means (although this is always an empirical question)
- Limitations for most of the ACS models:
  - Require **equal-interval** occasions (if they use the idea of "time lag")
  - Require **balanced** time across persons (no intermediate time values)
  - But **do not require complete data** (unlike when CS and UN are estimated via least squares in ANOVA instead of ML/REML in MLM)
- ACS models do require some new terminology to introduce...



# Two Families of ACS Models

- So far, we've referred to the variance and covariance matrix of the multivariate (longitudinal) outcomes as the **R** matrix
  - We now refer to these as "**R-only models**" (use **REPEATED** statement only)
  - Although the **R** matrix is actually specified per individual, ACS models usually assume the same **R** matrix for everyone
  - **R** matrix is symmetric with dimensions  $n \times n$ , in which  $n = \#$  occasions per person (although people can have missing data, the same set of *possible* occasions is required across people to use most **R-only** models)
- **3 other matrices we'll see in "G and R combined" ACS models:**
  - **G** = matrix of random effects variances and covariances (stay tuned)
  - **Z** = matrix of values for predictors that have random effects (stay tuned)
  - **V** = symmetric  $n \times n$  matrix of **total** variance and covariance over time
    - If the model includes random effects, then **G** and **Z** get combined with **R** to make **V** as  $\mathbf{V} = \mathbf{ZGZ}^T + \mathbf{R}$  (accomplished by adding the **RANDOM** statement)
    - If the model does NOT include random effects in **G**, then  $\mathbf{V} = \mathbf{R}$ ... so, **R-only**

# Review: Covariances and Correlations

$$\text{Correlation}_{y1,y2} = \frac{\text{Covariance}_{y1,y2}}{\sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}}$$

$$\text{Covariance}_{y1,y2} = \text{Correlation}_{y1,y2} * \sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}$$

- Given the standard deviation (as  $\sqrt{\text{Variance}}$ ) at each occasion, either the correlation and covariance can be calculated given the other
- ACS models with **homogeneous variances** tend to be specified in terms of **variance and covariance**
  - Given same variance over time, same covariance → same correlation
- ACS models with **heterogeneous variance** tend to be specified in terms of **variance and correlation**
  - Different variances over time → different covariances over time, even if the correlation is the same (so only correlation is estimated directly)

# R-Only ACS Models

- The **R-only** models to be presented next are all specified using the **REPEATED** statement only (no RANDOM statement)
- They are explained by showing their predicted **R** matrix, which provides the **total** variances and covariances across occasions
  - Total variance per occasion on diagonal
  - Total covariances across occasions on off-diagonals
  - I've included in " " the labels SAS uses for each parameter
- Correlations across occasions can be calculated given variances and covariances, which would be shown in the **RCORR** matrix (available in SAS PROC MIXED)
  - 1's on diagonal (standardized variables), correlations on off-diagonal
- **Unstructured (TYPE=UN) will always fit best by -2LL**
  - All ACS models are nested within Unstructured (UN = the data)
  - Goal: find an ACS model that is **simpler** but **not worse fitting** than UN

# R-Only ACS Models: CS/CSH

## • Compound Symmetry: TYPE=CS

➤ 2 parameters:

- 1 "residual" variance  $\sigma_e^2$
- 1 "CS" covariance across occasions

$$\begin{bmatrix} CS + \sigma_e^2 & CS & CS & CS \\ CS & CS + \sigma_e^2 & CS & CS \\ CS & CS & CS + \sigma_e^2 & CS \\ CS & CS & CS & CS + \sigma_e^2 \end{bmatrix}$$

➤ Constant total variance:  $CS + \sigma_e^2$

➤ Constant total covariance: CS

## • Compound Symmetry Heterogeneous: TYPE=CSH

➤  $n+1$  parameters:

- $n$  separate "Var( $n$ )" total variances  $\sigma_{Tn}^2$
- 1 "CSH" total correlation across occasions

$$\begin{bmatrix} \sigma_{T1}^2 & CSH\sigma_{T1}\sigma_{T2} & CSH\sigma_{T1}\sigma_{T3} & CSH\sigma_{T1}\sigma_{T4} \\ CSH\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & CSH\sigma_{T2}\sigma_{T3} & CSH\sigma_{T2}\sigma_{T4} \\ CSH\sigma_{T3}\sigma_{T1} & CSH\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & CSH\sigma_{T3}\sigma_{T4} \\ CSH\sigma_{T4}\sigma_{T1} & CSH\sigma_{T4}\sigma_{T2} & CSH\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

➤ Separate total variances are estimated directly

➤ Still constant total correlation: CSH (but has non-constant covariances)

# R-Only ACS Models: AR1/ARH1

## • 1<sup>st</sup> Order Auto-Regressive: TYPE=AR(1)

➤ 2 parameters:

- **1 constant total variance**  
 $\sigma_T^2$  (misabeled "residual")
- **1 "AR1" total auto-correlation**  $r_T$   
across occasions
- $r_T^1$  is lag-1 correlation,  $r_T^2$  is lag-2 correlation,  $r_T^3$  is lag-3 correlation....

$$\begin{bmatrix} \sigma_T^2 & r_T^1 \sigma_T^2 & r_T^2 \sigma_T^2 & r_T^3 \sigma_T^2 \\ r_T^1 \sigma_T^2 & \sigma_T^2 & r_T^1 \sigma_T^2 & r_T^2 \sigma_T^2 \\ r_T^2 \sigma_T^2 & r_T^1 \sigma_T^2 & \sigma_T^2 & r_T^1 \sigma_T^2 \\ r_T^3 \sigma_T^2 & r_T^2 \sigma_T^2 & r_T^1 \sigma_T^2 & \sigma_T^2 \end{bmatrix}$$

## • 1<sup>st</sup> Order Auto-Regressive Heterogeneous: TYPE=ARH(1)

➤  $n+1$  parameters:

- **$n$  separate "Var( $n$ )"**  
**total variances**  $\sigma_{Tn}^2$
- **1 "ARH1" total auto-**  
**correlation**  $r_T$  across occasions
- $r_T^1$  is lag-1 correlation,  $r_T^2$  is lag-2 correlation,  $r_T^3$  is lag-3 correlation....

$$\begin{bmatrix} \sigma_{T1}^2 & r_T^1 \sigma_{T1} \sigma_{T2} & r_T^2 \sigma_{T1} \sigma_{T3} & r_T^3 \sigma_{T1} \sigma_{T4} \\ r_T^1 \sigma_{T2} \sigma_{T1} & \sigma_{T2}^2 & r_T^1 \sigma_{T2} \sigma_{T3} & r_T^2 \sigma_{T2} \sigma_{T4} \\ r_T^2 \sigma_{T3} \sigma_{T1} & r_T^1 \sigma_{T3} \sigma_{T2} & \sigma_{T3}^2 & r_T^1 \sigma_{T3} \sigma_{T4} \\ r_T^3 \sigma_{T4} \sigma_{T1} & r_T^2 \sigma_{T4} \sigma_{T2} & r_T^1 \sigma_{T4} \sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

# R-Only ACS Models: TOEP<sub>n</sub>/TOEPH<sub>n</sub>

## • Toeplitz(*n*): TYPE=TOEP(*n*)

➤ *n* parameters:

- **1 constant total variance**  
 $\sigma_T^2$  (misabeled "residual")
- ***n* - 1 "TOEP(lag)"  $c_{Tn}$  banded total covariances** across occasions
- $c_{T1}$  is lag-1 covariance,  $c_{T2}$  is lag-2 covariance,  $c_{T3}$  is lag-3 covariance....

$$\begin{bmatrix} \sigma_T^2 & & & \\ c_{T1} & \sigma_T^2 & & \\ c_{T2} & c_{T1} & \sigma_T^2 & \\ c_{T3} & c_{T2} & c_{T1} & \sigma_T^2 \end{bmatrix}$$

## • Toeplitz Heterogeneous(*n*): TYPE=TOEPH(*n*)

➤ *n* + (*n* - 1) parameters:

- ***n* separate "Var(*n*)" total variances  $\sigma_{Tn}^2$**
- ***n* - 1 "TOEPH(lag)"  $r_{Tn}$  banded total correlations** across occasions

$$\begin{bmatrix} \sigma_{T1}^2 & r_{T1}\sigma_{T1}\sigma_{T2} & r_{T2}\sigma_{T1}\sigma_{T3} & r_{T3}\sigma_{T1}\sigma_{T4} \\ r_{T1}\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & r_{T1}\sigma_{T2}\sigma_{T3} & r_{T2}\sigma_{T2}\sigma_{T4} \\ r_{T2}\sigma_{T3}\sigma_{T1} & r_{T1}\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & r_{T1}\sigma_{T3}\sigma_{T4} \\ r_{T3}\sigma_{T4}\sigma_{T1} & r_{T2}\sigma_{T4}\sigma_{T2} & r_{T1}\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

- $r_{T1}$  is lag-1 correlation,  $r_{T2}$  is lag-2 correlation,  $r_{T3}$  is lag-3 correlation....

# Comparing **R**-only ACS Models

- Baseline models: **CS = simplest, UN = most complex**
  - Relative to CS, more complex models fit “better” or “not better”
  - Relative to UN, less complex models fit “worse” or “not worse”
- Other rules of nesting and model comparisons:
  - Homogeneous variance models are nested within heterogeneous variance models (e.g., CS in CSH, AR1 in ARH1, TOEP in TOEPH)
  - CS and AR1 are each nested within TOEP (i.e., TOEP can become CS or AR1 through restrictions of its covariance patterns)
  - CS and AR1 are not nested (because both have 2 parameters)
  - **R**-only models differ in unbounded parameters, so can be compared using regular  $-2\Delta LL$  tests (instead of mixture  $-2\Delta LL$  tests)
  - Good idea to start by assuming heterogeneous variances until you settle on the covariance pattern, then test if het. var. are still necessary
  - When in doubt, just compare AIC and BIC (useful even with  $-2\Delta LL$  tests)

# The Other Family of ACS Models

- **R**-only models *directly* predict the **total** variance and covariance
- **G** and **R** models *indirectly* predict the total variance and covariance through **between-person (BP)** and **within-person (WP)** sources of variance and covariance → So, for this model:  $\mathbf{y}_{ti} = \beta_0 + \mathbf{U}_{0i} + \mathbf{e}_{ti}$ 
  - **BP** = **G** matrix of **level-2 random effect ( $\mathbf{U}_{0i}$ )** variances and covariances
    - Which effects get to be random (whose variance and covariances are then included in **G**) is specified using the **RANDOM** statement (always TYPE=UN)
    - Our ACS models have a random intercept only, so **G** is 1x1 scalar of  $[\tau_{U_0}^2]$
  - **WP** = **R** matrix of **level-1 ( $\mathbf{e}_{ti}$ ) residual** variances and covariances
    - The  $n \times n$  **R** matrix of **residual** variances and covariances **that remain** after controlling for random intercept variance is then modeled with **REPEATED**
  - **Total** = **V** =  $n \times n$  matrix of **total** variance and covariance over time that results from putting **G** and **R** together:  $\mathbf{V} = \mathbf{ZGZ}^T + \mathbf{R}$ 
    - **Z** is a matrix that holds the values of predictors with random effects, but **Z** will be an  $n \times 1$  column of 1's for now (random intercept only)



# A “Random Intercept” (G and R) Model

Total Predicted Data Matrix is called **V Matrix**

$$\begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

## Level 2, BP Variance

Unstructured **G Matrix**  
(RANDOM statement)

Each person has same **1 x 1 G** matrix (no covariance across persons in two-level model)

Random Intercept Variance only

$$\begin{bmatrix} \tau_{U_0}^2 \end{bmatrix}$$

## Level 1, WP Variance

Diagonal (VC) **R Matrix**  
(REPEATED statement)

Each person has same **n x n R** matrix → **equal variances and 0 covariances** across time (no covariance across persons)

Residual Variance only

$$\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$$

# CS as a “Random Intercept” Model

**RI and DIAG:** Total predicted data matrix is called **V matrix**, created from the **G [TYPE=UN]** and **R [TYPE=VC]** matrices as follows:

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**Does the end result V look familiar? It should: CS =  $\tau_{U_0}^2$**

$$\begin{bmatrix} \text{CS} + \sigma_e^2 & \text{CS} & \text{CS} & \text{CS} \\ \text{CS} & \text{CS} + \sigma_e^2 & \text{CS} & \text{CS} \\ \text{CS} & \text{CS} & \text{CS} + \sigma_e^2 & \text{CS} \\ \text{CS} & \text{CS} & \text{CS} & \text{CS} + \sigma_e^2 \end{bmatrix}$$

So if the **R-only CS model** (the simplest baseline) can be specified equivalently using **G and R**, can we do the same for the **R-only UN model** (the most complex baseline)?

Absolutely! ...*with one small catch*

# UN via a “Random Intercept” Model

**RI and UN $n-1$ : Total predicted data matrix is called **V matrix**, created from the **G [TYPE=UN]** and **R [TYPE=UN( $n-1$ )] matrices as follows:****

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & \sigma_{e12} & \sigma_{e13} & \textcircled{0} \\ \sigma_{e21} & \sigma_{e2}^2 & \sigma_{e23} & \sigma_{e24} \\ \sigma_{e31} & \sigma_{e32} & \sigma_{e3}^2 & \sigma_{e34} \\ \textcircled{0} & \sigma_{e42} & \sigma_{e43} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + \sigma_{e12} & \tau_{U_0}^2 + \sigma_{e13} & \textcircled{\tau_{U_0}^2} \\ \tau_{U_0}^2 + \sigma_{e21} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + \sigma_{e23} & \tau_{U_0}^2 + \sigma_{e24} \\ \tau_{U_0}^2 + \sigma_{e31} & \tau_{U_0}^2 + \sigma_{e32} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + \sigma_{e34} \\ \textcircled{\tau_{U_0}^2} & \tau_{U_0}^2 + \sigma_{e42} & \tau_{U_0}^2 + \sigma_{e43} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

This **RI and UN $n-1$  model** is equivalent to (makes same predictions as) the **R-only UN model**. But it shows the *residual* (not total) covariances.

Because we can't estimate all possible variances and covariances in the **R** matrix and also estimate the random intercept variance  $\tau_{U_0}^2$  in the **G** matrix, we are eliminating the last **R** matrix covariance by setting it to 0.

Accordingly, in the **RI and UN $n-1$  model**, the random intercept variance  $\tau_{U_0}^2$  takes on the value of the covariance for the first and last occasions.

# Rationale for **G** and **R** ACS models

- Modeling WP fluctuation traditionally involves using **R** only (no **G**)  
→ **Total** BP + WP variance described by just **R** matrix (so **R=V**)
  - Correlations would still be expected even at distant time lags because of constant individual differences (i.e., the BP random intercept)
  - Resulting **R**-only model may require lots of estimated parameters as a result e.g., 8 time points? Pry need a 7-lag Toeplitz(8) model
- **Why not take out the primary reason for the covariance across occasions (the random intercept variance) and see what's left?**
  - Random intercept variance  $\tau_{U_0}^2$  in **G** → control for person mean differences
  - THEN predict just the **residual** variance/covariance in **R**, not the **total**
  - Resulting model may be more parsimonious (e.g., maybe only lag1 or lag2 occasions are still related after removing  $\tau_{U_0}^2$  as a source of covariance)
  - Has the advantage of still distinguishing BP from WP variance (useful for descriptive purposes and for calculating effect sizes later)

# Random Intercept + Diagonal R Models

**RI and DIAG:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=VC]:

*homogeneous* residual variances; **no** residual covariances

**Same fit as  
R-only CS**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**RI and DIAGH:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=UN(1)]:

*heterogeneous* residual variances; **no** residual covariances

**NOT same fit  
as R-only CSH**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & 0 & 0 & 0 \\ 0 & \sigma_{e2}^2 & 0 & 0 \\ 0 & 0 & \sigma_{e3}^2 & 0 \\ 0 & 0 & 0 & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + AR I R Models

**RI and AR1:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=AR(1)]:

*homogeneous* residual variances; *auto-regressive lagged* residual covariances

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & r_e^1 \sigma_e^2 & r_e^2 \sigma_e^2 & r_e^3 \sigma_e^2 \\ r_e^1 \sigma_e^2 & \sigma_e^2 & r_e^1 \sigma_e^2 & r_e^2 \sigma_e^2 \\ r_e^2 \sigma_e^2 & r_e^1 \sigma_e^2 & \sigma_e^2 & r_e^1 \sigma_e^2 \\ r_e^3 \sigma_e^2 & r_e^2 \sigma_e^2 & r_e^1 \sigma_e^2 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^3 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^3 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**RI and ARH1:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=ARH(1)]:

*heterogeneous* residual variances; *auto-regressive lagged* residual covariances

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_e^1 \sigma_{e1} \sigma_{e2} & r_e^2 \sigma_{e1} \sigma_{e3} & r_e^3 \sigma_{e1} \sigma_{e4} \\ r_e^1 \sigma_{e2} \sigma_{e1} & \sigma_{e2}^2 & r_e^1 \sigma_{e2} \sigma_{e3} & r_e^2 \sigma_{e2} \sigma_{e4} \\ r_e^2 \sigma_{e3} \sigma_{e1} & r_e^1 \sigma_{e3} \sigma_{e2} & \sigma_{e3}^2 & r_e^1 \sigma_{e3} \sigma_{e4} \\ r_e^3 \sigma_{e4} \sigma_{e1} & r_e^2 \sigma_{e4} \sigma_{e2} & r_e^1 \sigma_{e4} \sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e1} \sigma_{e2} & \tau_{U_0}^2 + r_e^2 \sigma_{e1} \sigma_{e3} & \tau_{U_0}^2 + r_e^3 \sigma_{e1} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^1 \sigma_{e2} \sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e2} \sigma_{e3} & \tau_{U_0}^2 + r_e^2 \sigma_{e2} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^2 \sigma_{e3} \sigma_{e1} & \tau_{U_0}^2 + r_e^1 \sigma_{e3} \sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e3} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^3 \sigma_{e4} \sigma_{e1} & \tau_{U_0}^2 + r_e^2 \sigma_{e4} \sigma_{e2} & \tau_{U_0}^2 + r_e^1 \sigma_{e4} \sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + TOEP $n-1$ R Models

**RI and TOEP $n-1$ :**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=TOEP( $n-1$ )]:

*homogeneous* residual variances; *banded* residual covariances

Same fit as  
R-only TOEP( $n$ )

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & c_{e1} & c_{e2} & 0 \\ c_{e1} & \sigma_e^2 & c_{e1} & c_{e2} \\ c_{e2} & c_{e1} & \sigma_e^2 & c_{e1} \\ 0 & c_{e2} & c_{e1} & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 \\ \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + c_{e2} \\ \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} \\ \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

Because of  $\tau_{U_0}^2$ ,  
highest lag  
covariance in  $\mathbf{R}$   
must be set to  
0 for model to  
be identified

**RI and TOEPH $n-1$ :**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=TOEPH( $n-1$ )]:

*heterogeneous* residual variances; *banded* residual covariances

NOT same fit as  
R-only TOEPH( $n$ )

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_{e1}\sigma_{e1}\sigma_{e2} & r_{e2}\sigma_{e1}\sigma_{e3} & 0 \\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma_{e2}^2 & r_{e1}\sigma_{e2}\sigma_{e3} & r_{e2}\sigma_{e2}\sigma_{e4} \\ r_{e2}\sigma_{e3}\sigma_{e1} & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma_{e3}^2 & r_{e1}\sigma_{e3}\sigma_{e4} \\ 0 & r_{e2}\sigma_{e4}\sigma_{e2} & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e1}\sigma_{e2} & \tau_{U_0}^2 + r_{e2}\sigma_{e1}\sigma_{e3} & \tau_{U_0}^2 \\ \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e3} & \tau_{U_0}^2 + r_{e2}\sigma_{e2}\sigma_{e4} \\ \tau_{U_0}^2 + r_{e2}\sigma_{e3}\sigma_{e1} & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e4} \\ \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e2}\sigma_{e4}\sigma_{e2} & \tau_{U_0}^2 + r_{e1}\sigma_{e4}\sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + TOEP2 R Models

**RI and TOEP2:**  $V$  is created from  $G$  [TYPE=UN] and  $R$  [TYPE=TOEP(2)]:  
*homogeneous* residual variances; *banded* residual covariance at **lag1** only

$$V = Z * G * Z^T + R = V$$

$$V = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & c_{e1} & 0 & 0 \\ c_{e1} & \sigma_e^2 & c_{e1} & 0 \\ 0 & c_{e1} & \sigma_e^2 & c_{e1} \\ 0 & 0 & c_{e1} & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

Now we can test the need for residual covariances at higher lags

**RI and TOEPH2:**  $V$  is created from  $G$  [TYPE=UN] and  $R$  [TYPE=TOEPH(2)]:  
*heterogeneous* residual variances; *banded* residual covariance at **lag1** only

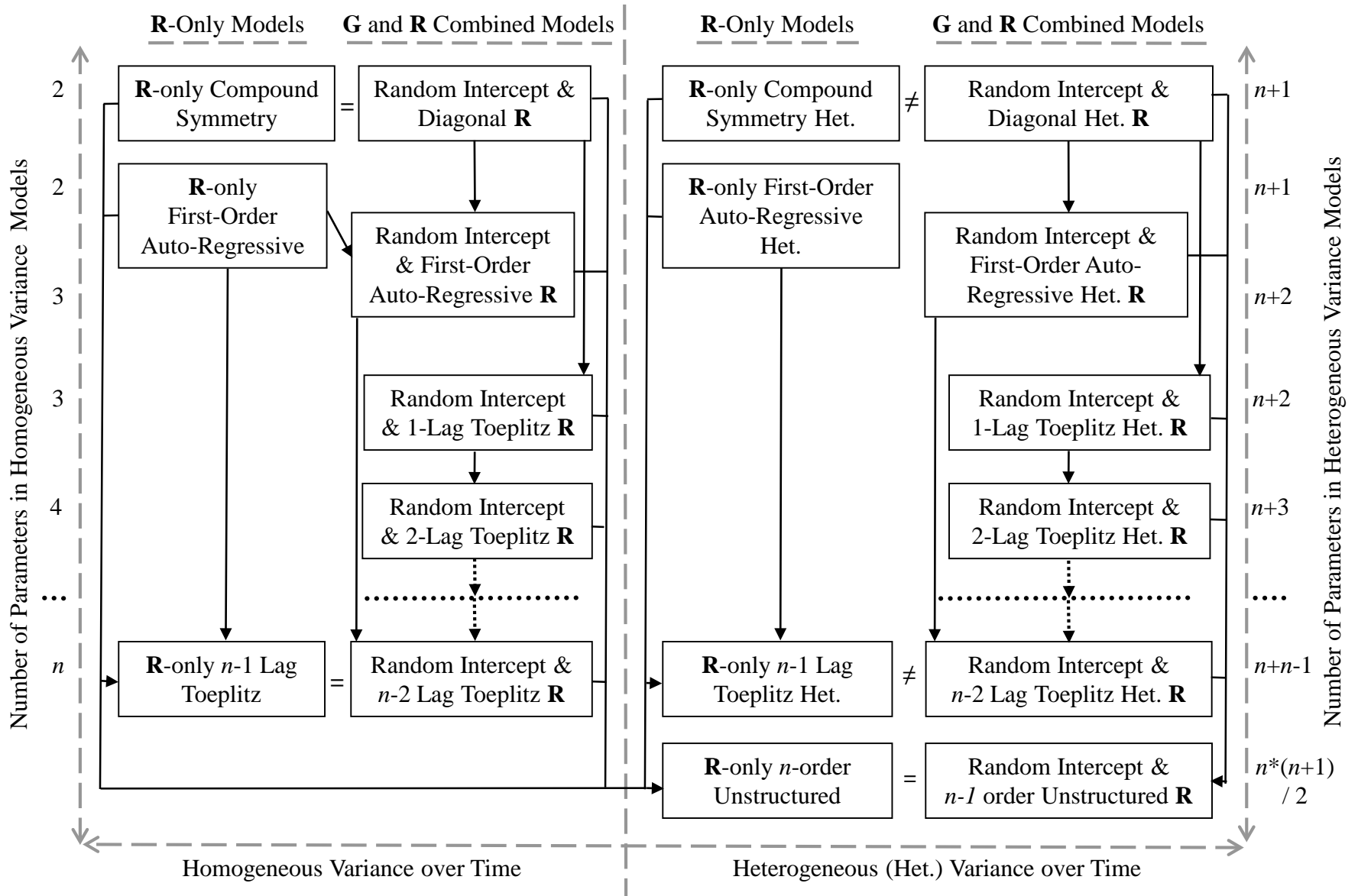
$$V = Z * G * Z^T + R = V$$

$$V = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_{e1}\sigma_{e1}\sigma_{e2} & 0 & 0 \\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma_{e2}^2 & r_{e1}\sigma_{e2}\sigma_{e3} & 0 \\ 0 & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma_{e3}^2 & r_{e1}\sigma_{e3}\sigma_{e4} \\ 0 & 0 & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e1}\sigma_{e2} & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e3} & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e4} \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e4}\sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$



# Map of **R**-only and **G** and **R** ACS Models

*Arrows indicate nesting (end is more complex model)*



# Stuff to Watch Out For...

- **If using a random intercept, don't forget to drop 1 parameter in:**
  - **$n-1$  order UN R:** Can't get all possible elements in **R**, plus  $\tau_{U_0}^2$  in **G**
  - **TOEP $n-1$ :** Have to eliminate last lag covariance
- If using a random intercept...
  - Can't do RI + CS **R**: Can't get a constant in **R**, and then another constant in **G**
  - Can often test if random intercept helps (e.g., AR1 is nested within RI + AR1)
- If "**time**" is treated as **continuous** in the fixed effects, you will need another variable for **time** that is **categorical** to use in the syntax:
  - "Continuous Time" → on MODEL statement
  - "Categorical Time" → on CLASS and REPEATED statements
- Most alternative covariance structure models assume **time is balanced across persons with equal intervals across occasions**
  - If not, holding correlations of same lag equal doesn't make sense
  - Other structures can be used for unbalanced time
    - SP(POW)(time) = AR1 for unbalanced time (see SAS REPEATED statement for others)

# Summary: Two Families of ACS Models

- **R**-only models:
  - Specify **R** model on REPEATED statement without any random effects variances in **G** (so no RANDOM statement is used)
  - Include UN, CS, CSH, AR1, AR1H, TOEP $n$ , TOEPH $n$  (among others)
  - *Total* variance and *total* covariance kept in **R**, so **R** = **V**
  - Other than CS, does not partition total variance into BP vs. WP
- **G** and **R** combined models (so **G** and **R**  $\rightarrow$  **V**):
  - Specify random intercept variance  $\tau_{U_0}^2$  in **G** using RANDOM statement, then specify **R** model using REPEATED statement
  - **G** matrix = Level-2 BP variance and covariance due to  $U_{0i}$ , so **R** = Level-1 WP variance and covariance of the  $e_{ti}$  residuals
  - **R** models what's left after accounting for mean differences between persons (via the random intercept variance  $\tau_{U_0}^2$  in **G**)

# Syntax for Models for the Variance

- Does your model include **random intercept variance**  $\tau_{U_0}^2$  (for  $U_{0i}$ ) ?
  - Use the **RANDOM** statement → **G matrix**
  - Random intercept models BP interindividual differences in mean Y
- What about **residual variance**  $\sigma_e^2$  (for  $e_{ti}$ ) ?
  - Use the **REPEATED** statement → **R matrix**
    - **WITHOUT a RANDOM statement: R is BP and WP variance together** =  $\sigma_T^2$   
→ Total variances and covariances (to model all variation, so **R = V**)
    - **WITH a RANDOM statement: R is WP variance only** =  $\sigma_e^2$   
→ Residual variances and covariances to model WP intraindividual variation  
→ **G** and **R** put back together = **V matrix** of total variances and covariances
- The **REPEATED** statement is always there implicitly...
  - Any model **always** has at least one residual variance in **R** matrix
- But the **RANDOM** statement is only there if you write it
  - **G** matrix isn't always necessary (don't always need random intercept)

# Wrapping Up: ACS Models

- Even if you just expect fluctuation over time rather than change, you still should be concerned about accurately predicting the variances and covariances across occasions
- Baseline models (from ANOVA least squares) are CS & UN:
  - Compound Symmetry: Equal variance and covariance over time
  - Unstructured: All variances & covariances estimated separately
  - CS and UN via ML or REML estimation allows missing data
- MLM gives us choices in the middle
  - Goal: Get as close to UN as parsimoniously as possible
  - **R**-only: Structure TOTAL variation in one matrix (**R** only)
  - **G**+**R**: Put constant covariance due to random intercept in **G**, then structural RESIDUAL covariance in **R** (so that **G** and **R** → **V** TOTAL)