A Crash Course in Multilevel Models for Longitudinal Data: Getting the model for "time" right

- Topics:
  - The Big Picture
  - ACS models using the **R** matrix only
  - Introducing the **G**, **Z**, and **V** matrices
  - ACS models combining the **G** and **R** matrices

Review of Steps in "Unconditional" Longitudinal Modeling (→ Time only)

For all outcomes:

1. Empty Model; Calculate ICC
2. Decide on a metric of time
3. Decide on a centering point
4. Estimate means model and plot individual trajectories

If your outcome shows systematic change:

5. Evaluate fixed and random effects of time
6. Still consider possible alternative models for the residuals (**R** matrix)

If your outcome does NOT show ANY systematic change:

5. Evaluate alternative models for the variances (**G**+**R**, or **R**)

1. Empty Means, Random Intercept Model

- Not really predictive, but is a useful parsimonious baseline
  - Fit of "worst" longitudinal model to start building from
  - Partitions variance into between- and within-person variance

- Calculate **ICC** = between / (between + within variance)
  - = Average correlation between occasions
  - = Proportion of variance that is between persons
  - Effect size for amount of person dependency due to mean differences

- Tells you where the action will be:
  - If most of the variance is **between-persons in the random intercept (at level 2)**, you will need **person-level** predictors to reduce that variance (i.e., to account for inter-individual differences)
  - If most of the variance is **within-persons in the residual (at level 1)**, you will need **time-level** predictors to reduce that variance (i.e., to account for intra-individual differences)

2. Decide on the Metric of Time

- "Occasion of Study" as Time:
  - Can be used generically for many purposes—is my preferred default
  - Can still include age, event time as time-invariant predictors of change

- "Age" as Time:
  - Is equivalent to time-in-study if same age at beginning of study
  - Implies age convergence → that people only differ in age regardless of when they came into the study (BP effects = WP effects)

- "Distance to/from an Event" as Time:
  - Is appropriate if a distinct process is responsible for changes
  - Also implies convergence (BP effects = WP effects)
  - Only includes people that have experienced the event

- Make sure to use exact time regardless of which "time" used

3. Decide on a Centering Point

- How to choose: At what occasion would you like a snap-shot of inter-individual differences?
  - Intercept variance represents inter-individual differences at that particular time point (that you can later predict!)

- Where do you want your intercept?
  - Re-code time such that the centering point = 0
  - Use ESTIMATE statements to get predictions at other times

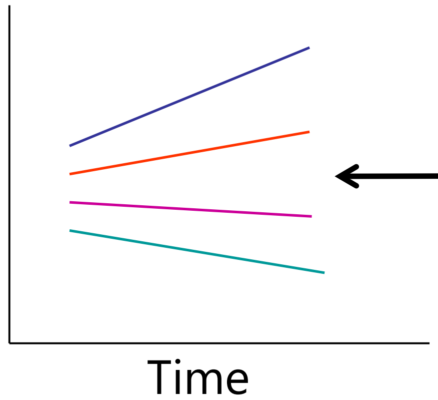
- Different versions of time = 0 will produce statistically equivalent models with re-arranged parameters
  - i.e., conditional level and rate of change at time 0

4. Plot Saturated Means and Individuals

- If time is <u>balanced</u> across persons:
  - Estimate a saturated means model to generate means
- If time is <u>NOT balanced</u> across persons:
  - Create a rounded time variable to estimate means model ONLY
  - Still use exact time/age variable for analysis!
- Plot the means – what kind of trajectory do you see?
- Please note: ML/REML estimated means per occasion may NOT be the same as the observed means (i.e., as given by PROC MEANS). The estimated means are what would have been obtained *had your data been complete* (assuming MAR), whereas observed means are not adjusted to reflect any missing data (MCAR). Report the ML/REML estimated means.

Modeling Change vs. Fluctuation

Pure WP Change



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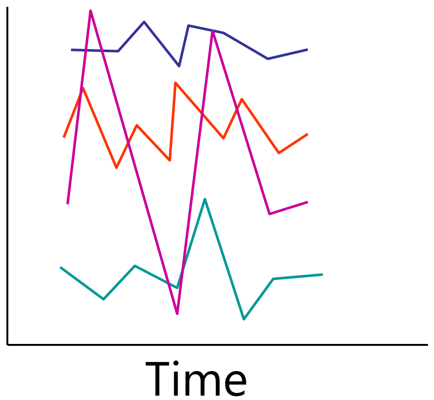
Pure WP Fluctuation



Model for the Means:

- WP Change → describe pattern of *average* change (over "time")
- **WP Fluctuation** → *may* not need anything (if no systematic change)

Model for the Variance:

- WP Change → describe *individual differences* in change (random effects)
  → this allows variances and covariances to differ over time
- **WP Fluctuation** → describe pattern of variances and covariances over time

Big Picture Framework: Models for the Variance in Longitudinal Data

Compound Symmetry (CS)

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

Most useful model: likely somewhere in between!

Unstructured (UN)

$$\begin{bmatrix} \sigma^2_{T1} & \sigma_{T12} & \sigma_{T13} & \sigma_{T14} \\ \sigma_{T21} & \sigma^2_{T2} & \sigma_{T23} & \sigma_{T24} \\ \sigma_{T31} & \sigma_{T32} & \sigma^2_{T3} & \sigma_{T43} \\ \sigma_{T41} & \sigma_{T42} & \sigma_{T43} & \sigma^2_{T4} \end{bmatrix}$$

Parsimony ⟵⟶ **Good fit**

Univariate RM ANOVA

NAME ...THAT ... STRUCTURE!

Multivariate RM ANOVA

What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Alternative Covariance Structure Models

- Useful in predicting patterns of variance and covariance that arise from **fluctuation in the outcome** across time:
  - **Variances**: Same (homogeneous) or different (heterogeneous)?
  - **Covariances**: Same or different? If different, what is the pattern?
    - Models with heterogeneous variances predict correlation instead of covariance because covariances will differ when variances differ
  - Often don't need any fixed effects for systematic effects of time in the model for the means (although this is always an empirical question)

- Limitations for most of the ACS models:
  - Require **equal-interval** occasions (if they use the idea of "time lag")
  - Require **balanced** time across persons (no intermediate time values)
  - But **do not require complete data** (unlike when CS and UN are estimated via least squares in ANOVA instead of ML/REML in MLM)

- ACS models do require some new terminology to introduce…

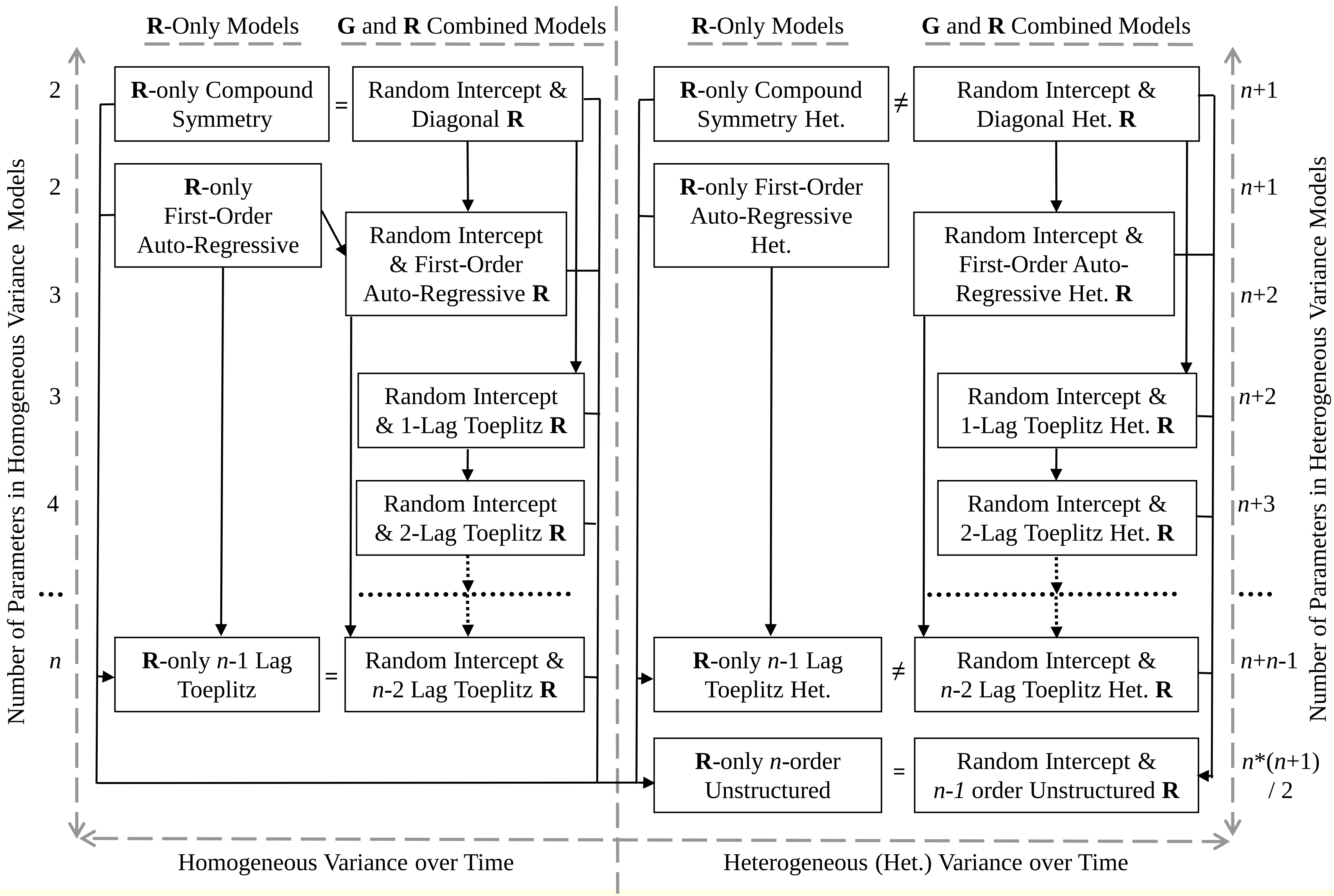
Two Families of ACS Models (ch. 4)

- **R**-only models:
  - Specify **R** model on REPEATED statement without any random effects variances in **G** (so no RANDOM statement is used)
  - Include UN, CS, CSH, AR1, AR1H, TOEP*n*, TOEPH*n* (among others)
  - *Total* variance and *total* covariance kept in **R**, so **R** = **V**
  - Other than CS, does not partition total variance into BP vs. WP

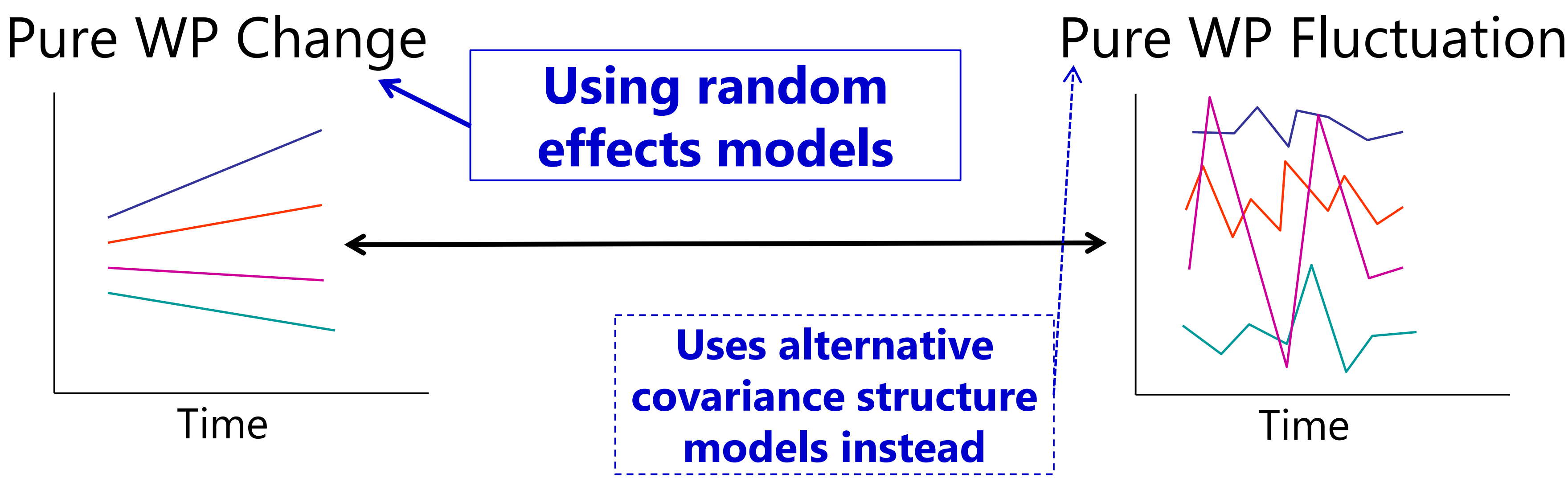
- **G** and **R** combined models (so **G** and **R** → **V**):
  - Specify random intercept variance $\tau^2_{U_0}$ in **G** using RANDOM statement, then specify **R** model using REPEATED statement
  - **G** matrix = Level-2 BP variance and covariance due to U_{0i}, so **R** = Level-1 WP variance and covariance of the e_{ti} residuals
  - **R** models what's left after accounting for mean differences between persons (via the random intercept variance $\tau^2_{U_0}$ in **G**)

Map of **R**-only and **G** and **R** ACS Models

Arrows indicate nesting (end is more complex model)



Modeling Change vs. Fluctuation



Model for the Means:

- **WP Change** → describe pattern of *average* change (over "time")
- WP Fluctuation → *may* not need anything (if no systematic change)

Model for the Variance:

- **WP Change** → describe *individual differences* in change (random effects)
  → this allows variances and covariances to differ over time
- WP Fluctuation → describe pattern of variances and covariances over time

The Big Picture of Longitudinal Data: **Models for the Means**

- What kind of change occurs on average over "time"? There are two baseline models to consider:
  - "**Empty**" → only a fixed intercept (predicts no change)
  - "**Saturated**" → all occasion mean differences from time 0 (ANOVA model that uses # fixed effects= *n*) *** *may not be possible in unbalanced data*

Parsimony ← **Empty Model:** Predicts NO change over time — **1 Fixed Effect**

In-between options: polynomial slopes, piecewise slopes, nonlinear slopes…

Saturated Means: Reproduces mean at each occasion — **# Fixed Effects = # Occasions** → **Good fit**

Name… that… Trajectory!

The Big Picture of Longitudinal Data:
Models for the Variance

<u>Compound Symmetry (CS)</u>

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

Most useful model: likely somewhere in between!

<u>Unstructured (UN)</u>

$$\begin{bmatrix} \sigma^2_{T1} & \sigma_{T12} & \sigma_{T13} & \sigma_{T14} \\ \sigma_{T21} & \sigma^2_{T2} & \sigma_{T23} & \sigma_{T24} \\ \sigma_{T31} & \sigma_{T32} & \sigma^2_{T3} & \sigma_{T43} \\ \sigma_{T41} & \sigma_{T42} & \sigma_{T43} & \sigma^2_{T4} \end{bmatrix}$$

Parsimony ← → **Good fit**

Univariate RM ANOVA

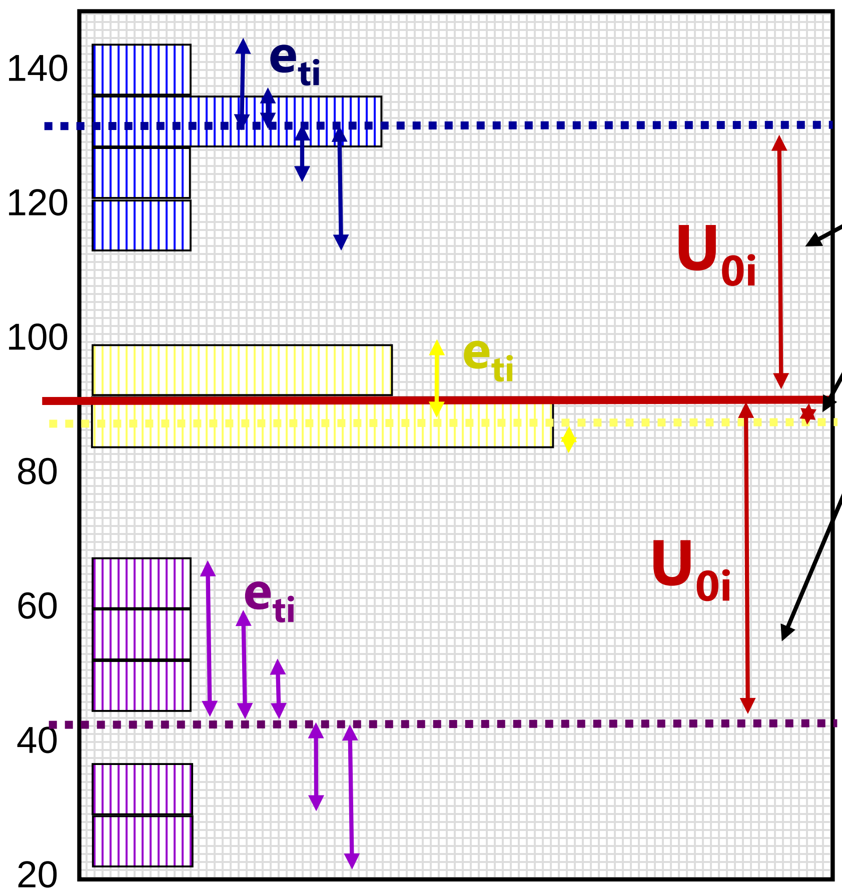
Name …that … Structure!

Multivariate RM ANOVA

What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Empty +Within-Person Model



y_{ti} variance → 2 sources:

Level 2 Random Intercept Variance **(of U_{0i}, as $\tau^2_{U_0}$):**

→ **Between**-Person Variance

→ Differences from **GRAND** mean

→ **INTER**-Individual Differences

Level 1 Residual Variance **(of e_{ti}, as σ^2_e):**

→ **Within**-Person Variance

→ Differences from **OWN** mean

→ **INTRA**-Individual Differences

Empty Means, Random Intercept Model

GLM Empty Model:

- $y_i = \beta_0 + e_i$

MLM Empty Model:

- Level 1:

  $y_{ti} = \beta_{0i} + e_{ti}$

- Level 2:

  $\beta_{0i} = \gamma_{00} + U_{0i}$

Fixed Intercept = mean of means (=mean because no predictors yet)

Random Intercept = individual-specific deviation from predicted intercept

Residual = time-specific deviation from individual's predicted outcome

3 Parameters:

Model for the Means (1):

- Fixed Intercept γ_{00}

Model for the Variance (2):

- Level-1 Variance of e_{ti} → σ_e^2
- Level-2 Variance of U_{0i} → $\tau_{U_0}^2$

Composite equation:

$y_{ti} = (\gamma_{00} + U_{0i}) + e_{ti}$

Augmenting the empty means, random intercept model with *time*

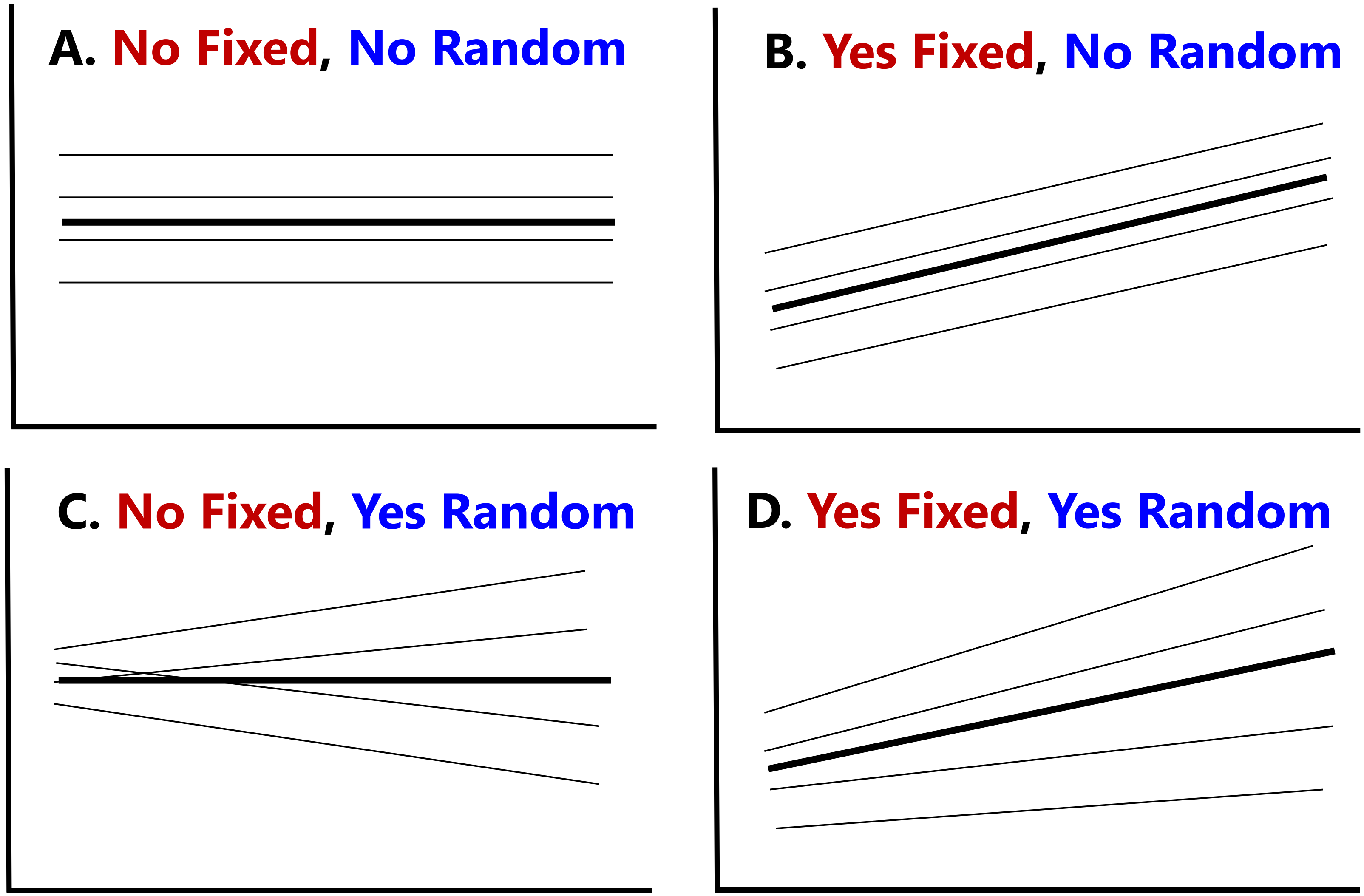
- 2 questions about the possible effects of *time*:

1. **Is there an effect of time on average?**
   - Is the line describing the sample means not flat?
   - Significant **FIXED** effect of time

2. **Does the average effect of time vary across individuals?**
   - Does each individual need his or her own line?
   - Significant **RANDOM** effect of time

Fixed and Random Effects of **Time**

(Note: The intercept is random in every figure)



B. Fixed Linear Time, Random Intercept Model
(4 parameters: effect of time is FIXED only)

Multilevel Model

Residual = time-specific deviation from individual's predicted outcome → estimated variance of σ_e^2

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(Time_{ti}) + e_{ti}$

Fixed Intercept = predicted mean outcome at time 0

Fixed Linear Time Slope = predicted mean rate of change per unit time

Level 2: $\beta_{0i} = \gamma_{00} + U_{0i}$ $\quad \beta_{1i} = \gamma_{10}$

Random Intercept = individual-specific deviation from fixed intercept → estimated variance of $\tau_{U_0}^2$

Composite Model

$$y_{ti} = \underbrace{(\gamma_{00} + U_{0i})}_{\beta_{0i}} + \underbrace{(\gamma_{10})}_{\beta_{1i}}(Time_{ti}) + e_{ti}$$

Because the effect of time is **fixed**, everyone is predicted to change at exactly the same rate.

C or D: Random Linear Time Model (6 parms)

Residual = time-specific deviation from individual's predicted outcome → estimated variance of σ^2_e

Multilevel Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}(Time_{ti}) + e_{ti}$

Fixed Intercept = predicted mean outcome at time 0

Fixed Linear Time Slope = predicted mean rate of change per unit time

Level 2: $\beta_{0i} = \gamma_{00} + U_{0i}$ $\quad \beta_{1i} = \gamma_{10} + U_{1i}$

Random Intercept = individual-specific deviation from fixed intercept at time 0 → estimated variance of $\tau^2_{U_0}$

Random Linear Time Slope = individual-specific deviation from fixed linear time slope → estimated variance of $\tau^2_{U_1}$

Also has an estimated covariance of random intercepts and slopes of $\tau_{U_{01}}$

Composite Model

$$y_{ti} = \underbrace{(\gamma_{00} + U_{0i})}_{\beta_{0i}} + \underbrace{(\gamma_{10} + U_{1i})}_{\beta_{1i}}(Time_{ti}) + e_{ti}$$

Random Linear Time Model
(6 parameters: effect of time is now RANDOM)

- Predicted total variances and covariances per person:

$$\mathbf{V}_i = \mathbf{Z}_i * \mathbf{G}_i * \mathbf{Z}_i^T + \mathbf{R}_i$$

$$\mathbf{V}_i = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \tau^2_{U_0} & \tau_{U_{01}} \\ \tau_{U_{01}} & \tau^2_{U_1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} \sigma^2_e & 0 & 0 & 0 \\ 0 & \sigma^2_e & 0 & 0 \\ 0 & 0 & \sigma^2_e & 0 \\ 0 & 0 & 0 & \sigma^2_e \end{bmatrix}$$

$\mathbf{V}_i$ matrix: Variance$\left[y_{time}\right]$

$$= \tau^2_{U_0} + \left[(time)^2 \tau^2_{U_1}\right] + \left[2(time)\tau_{U_{01}}\right] + \sigma^2_e$$

$\mathbf{V}_i$ matrix: Covariance$\left[y_A, y_B\right]$

$$= \tau^2_{U_0} + \left[(A+B)\tau_{U_{01}}\right] + \left[(AB)\tau^2_{U_1}\right]$$

$\mathbf{V}_i$ matrix = complicated ☺

$\mathbf{Z}_i$ = *n* x *u* values of **predictors with random effects**, so can differ per person (*u* = 2: int., time slope)

$\mathbf{Z}_i^T$ = *u* x *n* values of predictors with random effects (just **$\mathbf{Z}_i$** transposed)

$\mathbf{G}_i$ = *u* x *u* estimated **random effects variances and covariances**, so will be the same for all persons ($\tau^2_{U_0}$ = int. var., $\tau^2_{U_1}$ = slope var.)

$\mathbf{R}_i$ = *n* x *n* **time-specific residual variances and covariances**, so will be same for all persons (here, just diagonal σ^2_e)

G, R, and V: The Take-Home Point

- The partitioning of variance into piles…
  - **Level 2 = BP → G** matrix of random effects variances/covariances
  - **Level 1 = WP → R** matrix of residual variances/covariances
  - **G** and **R** combine via **Z** to create **V** matrix of total variances/covariances
  - Many flexible options that allows the variances and covariances to vary in a time-dependent way that better matches the actual data
    - Can allow differing variance and covariance due to other predictors, too

Compound Symmetry (CS)

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

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Random effects models use **G** and **R** to predict something in-between!

Parsimony ← → **Good fit**

Univariate RM ANOVA

Multivariate RM ANOVA

Summary: Modeling Means and Variances

- We have two tasks in describing within-person change:

- **Choose a Model for the Means**
  - What kind of change in the outcome do we have **on average**?
  - What kind and how many **fixed effects** do we need to predict that mean change as parsimoniously but accurately as possible?

- **Choose a Model for the Variance**
  - What pattern do the variances and covariances of the outcome show over time because of **individual differences** in change?
  - What kind and how many **random effects** do we need to predict that pattern as parsimoniously but accurately as possible?

The Big Picture of Longitudinal Data: Model for the Means (Fixed Effects)

- What kind of change occurs on average over "time"?
  - What is the most appropriate **metric of time**?
    - Time in study (with predictors for BP differences in time)?
    - Time since birth (age)? Time to event (time since diagnosis)?
    - Measurement occasions need not be the same across persons or equally spaced (code time as exactly as possible)
  - What kind of **theoretical process** generated the observed trajectories, and thus what kind of model do we need?
    - Linear or nonlinear? Continuous or discontinuous? Does change keep happening or does it eventually stop?
    - Many options: polynomial, piecewise, and nonlinear families

Name that trajectory… Polynomial?

- Predict **mean change** with **polynomial fixed effects of time**:
  - Linear → *constant* amount of change (up or down)
  - Quadratic → *change* in linear rate of change (acceleration/deceleration)
  - Cubic → *change* in acceleration/deceleration of linear rate of change (known in physics as jerk, surge, or jolt)
  - Terms work together to describe curved trajectories

- **Can have polynomial fixed time slopes UP TO: *n* – 1***
  - 3 occasions = 2nd order ($time^2$)= Fixed Quadratic Time or less
  - 4 occasions = 3rd order ($time^3$) = Fixed Cubic Time or less

- Interpretable polynomials past cubic are rarely seen in practice

*****$n-1$ rule can be broken in unbalanced data (but cautiously)

Interpreting Quadratic Fixed Effects

A Quadratic time effect is a two-way interaction: time*time

- Fixed quadratic time = "**half the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**

- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…

- The "twice" part comes from taking the derivatives of the function:

Intercept (Position) at Time T: $\hat{y}_T = 50.0 + 4.0T + 0.3T^2$

First Derivative (Velocity) at Time T: $\dfrac{d\hat{y}_T}{d(T)} = 4.0 + 0.6T$

Second Derivative (Acceleration) at Time T: $\dfrac{d^2\hat{y}_T}{d(T)} = 0.6$

Interpreting Quadratic Fixed Effects

A Quadratic time effect is a two-way interaction: time*time

- Fixed quadratic time = "**half** **the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**
- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…
- The "twice" part also comes from what you remember about the role of interactions with respect to their constituent main effects:

$$\hat{y} = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ$$

$$\text{Effect of X} = \beta_1 + \beta_3 Z$$

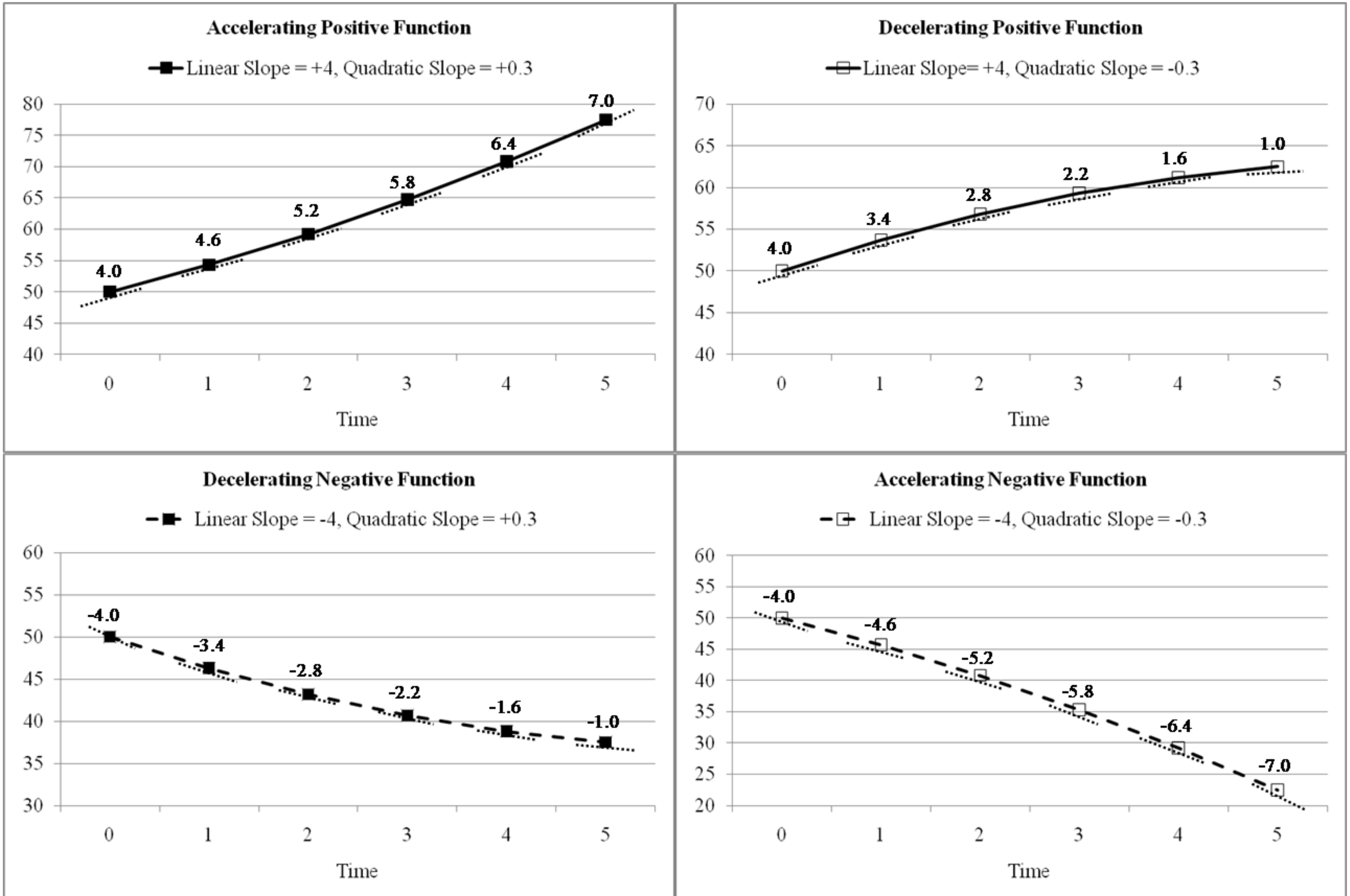
$$\text{Effect of Z} = \beta_2 + \beta_3 X$$

$$\hat{y}_T = \beta_0 + \beta_1 \text{Time}_T + \underline{\qquad} + \beta_3 \text{Time}_T^2$$

$$\text{Effect of Time}_T = \beta_1 + 2\beta_3 \text{Time}_T$$

- Because time is interacting with itself, there is no second main effect in the model for the interaction to modify as usual. So the quadratic time effect gets applied twice to the one (main) linear effect of time.

Examples of Fixed Quadratic Time Effects



Conditionality of Polynomial Fixed Time Effects

- We've seen how main effects become conditional simple effects once they are part of an interaction
- The same is true for polynomial **fixed effects of time**:
  - ***Fixed Intercept Only?***
    - Fixed Intercept = predicted mean of Y *for any occasion (= grand mean)*
  - ***Add Fixed Linear Time?***
    - Fixed Intercept = **now** predicted mean of Y from linear time *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Linear Time = mean linear rate of change *across all occasions* *(would be the same if time was centered elsewhere)*
  - ***Add Fixed Quadratic Time?***
    - Fixed Intercept = still predicted mean of Y *at time=0 (but from quadratic model)* (*would be different if time was centered elsewhere*)
    - Fixed Linear Time = **now** mean linear rate of change *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Quadratic Time = half the mean rate of acceleration or deceleration of change *across all occasions* (i.e., the linear slope changes the same over time)

Polynomial Fixed vs. Random Time Effects

- **Polynomial fixed effects** combine to describe mean trajectory over time (can have fixed slopes up to ***n* – 1**):
  - Fixed Intercept = Predicted mean level (at time 0)
  - Fixed Linear Time = Mean linear rate of change (at time 0)
  - Fixed Quadratic Time = Half of mean acceleration/deceleration in linear rate of change (2*quad is how the linear time slope changes per unit time if quadratic is highest order fixed effect of time)

- **Polynomial random effects** (individual deviations from the fixed effect) describe individual differences in those change parameters (can have random slopes up to ***n* – 2**):
  - Random Intercept = BP variance in level (at time 0)
  - Random Linear Time = BP variance in linear time slope (at time 0)
  - Random Quadratic Time = BP variance in half the rate of acceleration/deceleration of linear time slope (across all time if quadratic is highest-order random effect of time)

Random Quadratic Time Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

- β_{0i}: **Intercept** for person *i*
- γ_{00}: Fixed (mean) **Intercept**
- U_{0i}: Random (Deviation) **Intercept**

$\beta_{1i} = \gamma_{10} + U_{1i}$

- β_{1i}: **Linear Slope** for person *i*
- γ_{10}: Fixed (mean) **Linear Slope**
- U_{1i}: Random (Deviation) **Linear Slope**

$\beta_{2i} = \gamma_{20} + U_{2i}$

- β_{2i}: **Quad Slope** for person *i*
- γ_{20}: Fixed (mean) **Quad Slope**
- U_{2i}: Random (Deviation) **Quad Slope**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (n):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

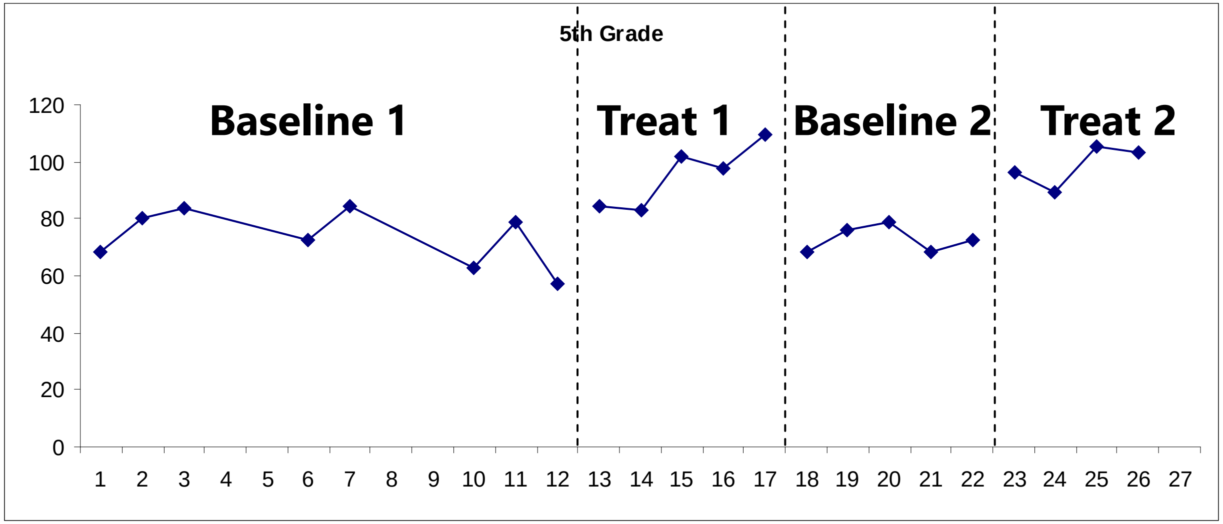
Need $n = 4$ occasions to fit random quadratic time model

Conditionality of Polynomial Random Effects

- We saw previously that lower-order fixed effects of time are conditional on higher-order polynomial fixed effects of time
- The same is true for polynomial **random effects of time**:
  - ***Random Intercept Only?***
    - Random Intercept = BP variance *for any occasion* in predicted mean Y (= variance in grand mean because individual lines are parallel)
  - ***Add Random Linear Time?***
    - Random Intercept = **now** BP variance *at time=0* in predicted mean Y *(would be different if time was centered elsewhere)*
    - Random Linear Time = BP variance *across all occasions* in linear rate of change *(would be the same if time was centered elsewhere)*
  - ***Add Random Quadratic Time?***
    - Random Intercept = still BP variance *at time=0* in predicted mean Y
    - Random Linear Time = **now** BP variance *at time=0* in linear rate of change *(would be different if time was centered elsewhere)*
    - Random Quadratic Time = BP variance *across all occasions* in half of accel/decel of change *(would be the same if time was centered elsewhere)*

Other Random Effects Models of Change

- **Piecewise models**: Discrete slopes for discrete phases of time
  - Separate terms describe sections of overall trajectories
  - Useful for examining change in intercepts and slopes before/after discrete events (changes in policy, interventions)
  - **Must know where the break point is ahead of time!**



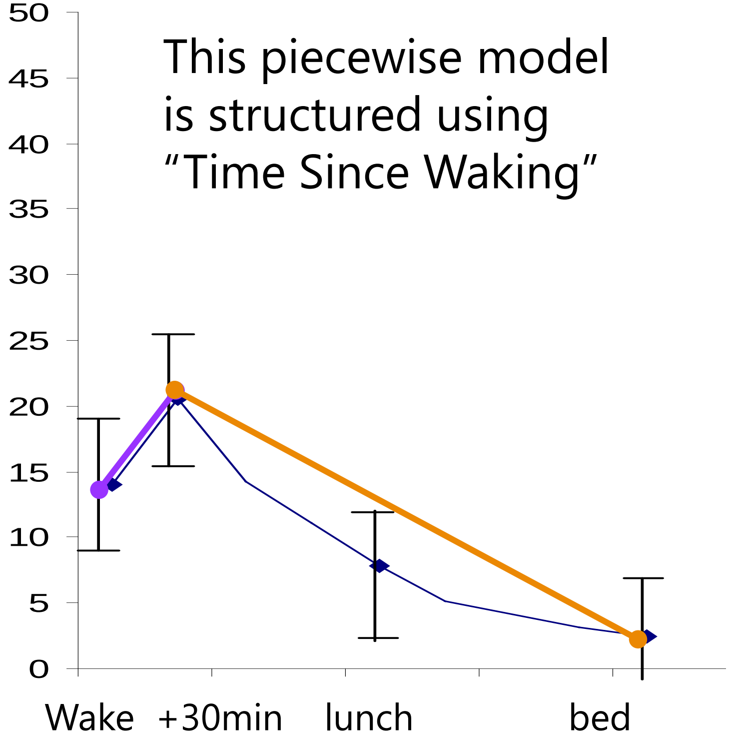
Piecewise Model:

4 slopes
(one per phase)

3 "jumps"
(shift in intercept between phases)

Example of Daily Cortisol Fluctuation: Morning Rise and Afternoon Decline

Average Trajectories

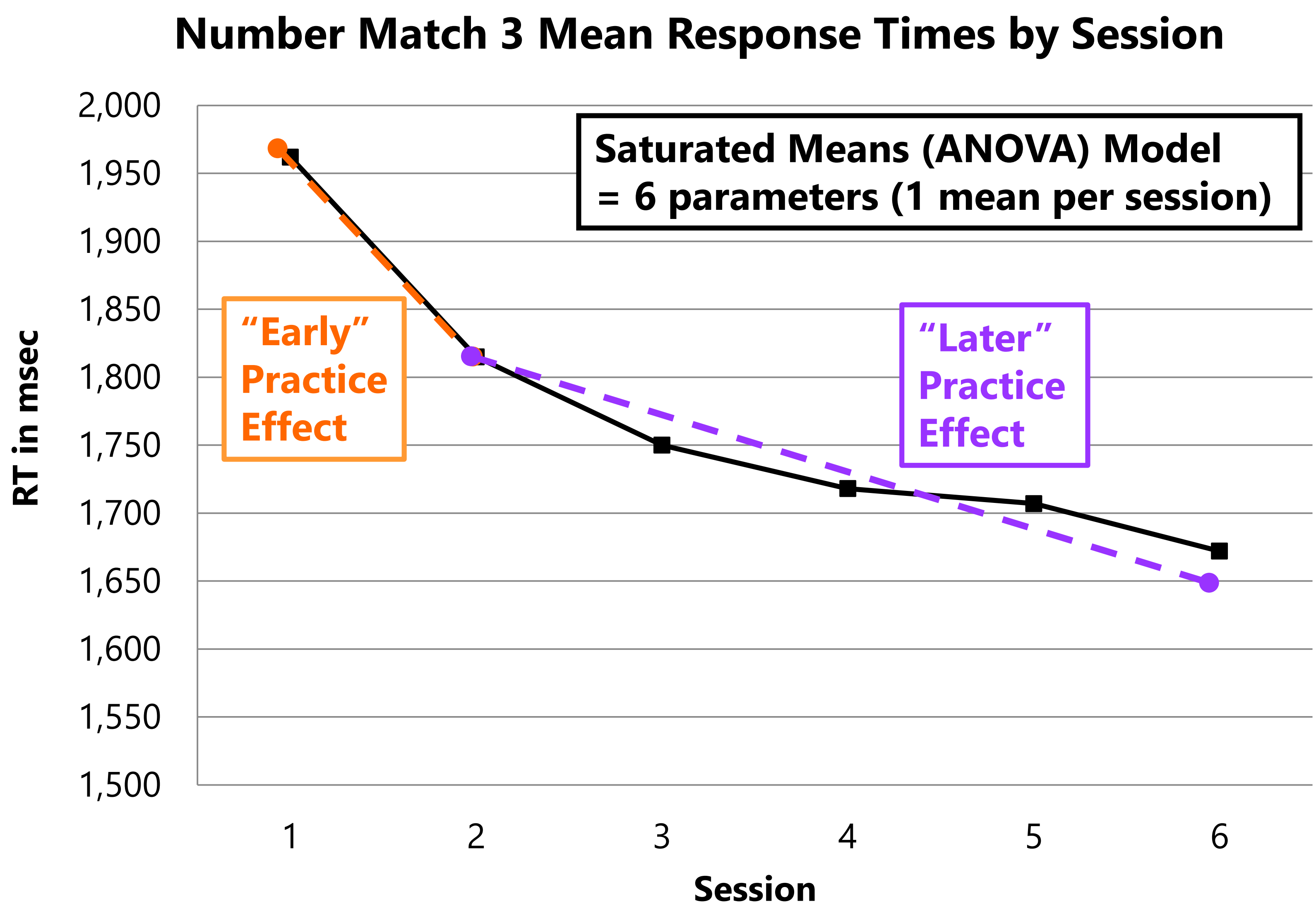


SAS Code to create two piecewise slopes from continuous time of day in stacked data:

```
IF occasion=1 THEN DO;
    P1=0;                P2=0; END;
IF occasion=2 THEN DO;
    P1= time2-time1; P2=0; END;
IF occasion=3 THEN DO;
    P1= time2-time1; P2=time3-time2; END;
IF occasion=4 THEN DO;
    P1= time2-time1; P2=time4-time2; END;
```

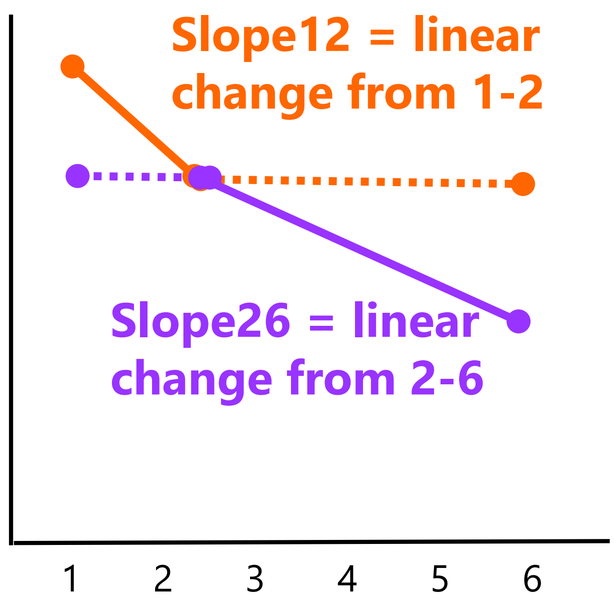
Note that a quadratic slope may be necessary for the afternoon decline slope!

What kind of piecewise model could predict our example data mean change across sessions?



Piecewise Models: Two Direct Slopes

- "Early Practice Slope" and "Later Practice Slope"
- Use to specify slopes through each discrete phase directly (can request test of difference)
- Session (1-6) gets recoded into 2 new time predictor variables, as shown below:



Session	1	2	3	4	5	6
Early Practice → Slope12 =	0	1	1	1	1	1
Later Practice → Slope26 =	0	0	1	2	3	4

2 Direct Slopes Model: Random Effects

- Parameters directly **represent each part** of trajectory:

- **Fixed effects** for mean change over time (3 fixed effects):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope12 = expected linear rate of change from 1 to 2
  - Fixed Slope26 = expected linear rate of change from 2 to 6

- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level
    when both slopes = 0 (at Session 1)
  - Random Slope12 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in linear slope from 2 to 6

Random Two-Slope Piecewise Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Slope1_{ti} + \beta_{2i}Slope2_{ti} + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

- β_{0i}: **Intercept** for person *i*
- γ_{00}: Fixed (mean) **Intercept**
- U_{0i}: Random (Deviation) **Intercept**

$\beta_{1i} = \gamma_{10} + U_{1i}$

- β_{1i}: **Slope1** for person *i*
- γ_{10}: Fixed (mean) **Slope1**
- U_{1i}: Random (Deviation) **Slope1**

$\beta_{2i} = \gamma_{20} + U_{2i}$

- β_{2i}: **Slope2** for person *i*
- γ_{20}: Fixed (mean) **Slope2**
- U_{2i}: Random (Deviation) **Slope2**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (*n*):

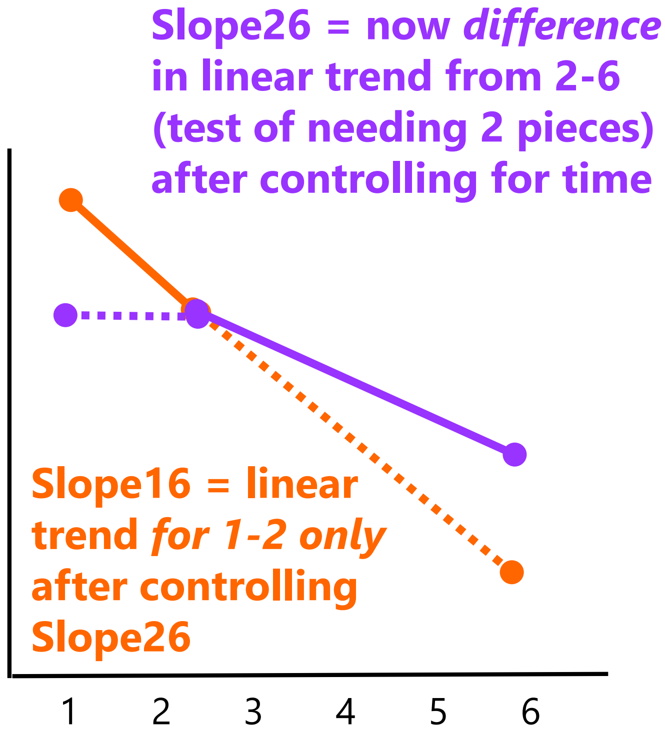
Fixed slopes = *n* – 1

Random slopes = *n* – 2

Need *n* = 4 occasions to fit random two-slope model

Piecewise Models: Slope + Deviation Slope

- "Linear Time Slope" and "Deviation Slope"
- Use to test if multiple slopes are needed directly in model
- Initial slope predictor is coded differently, second slope predictor is same:



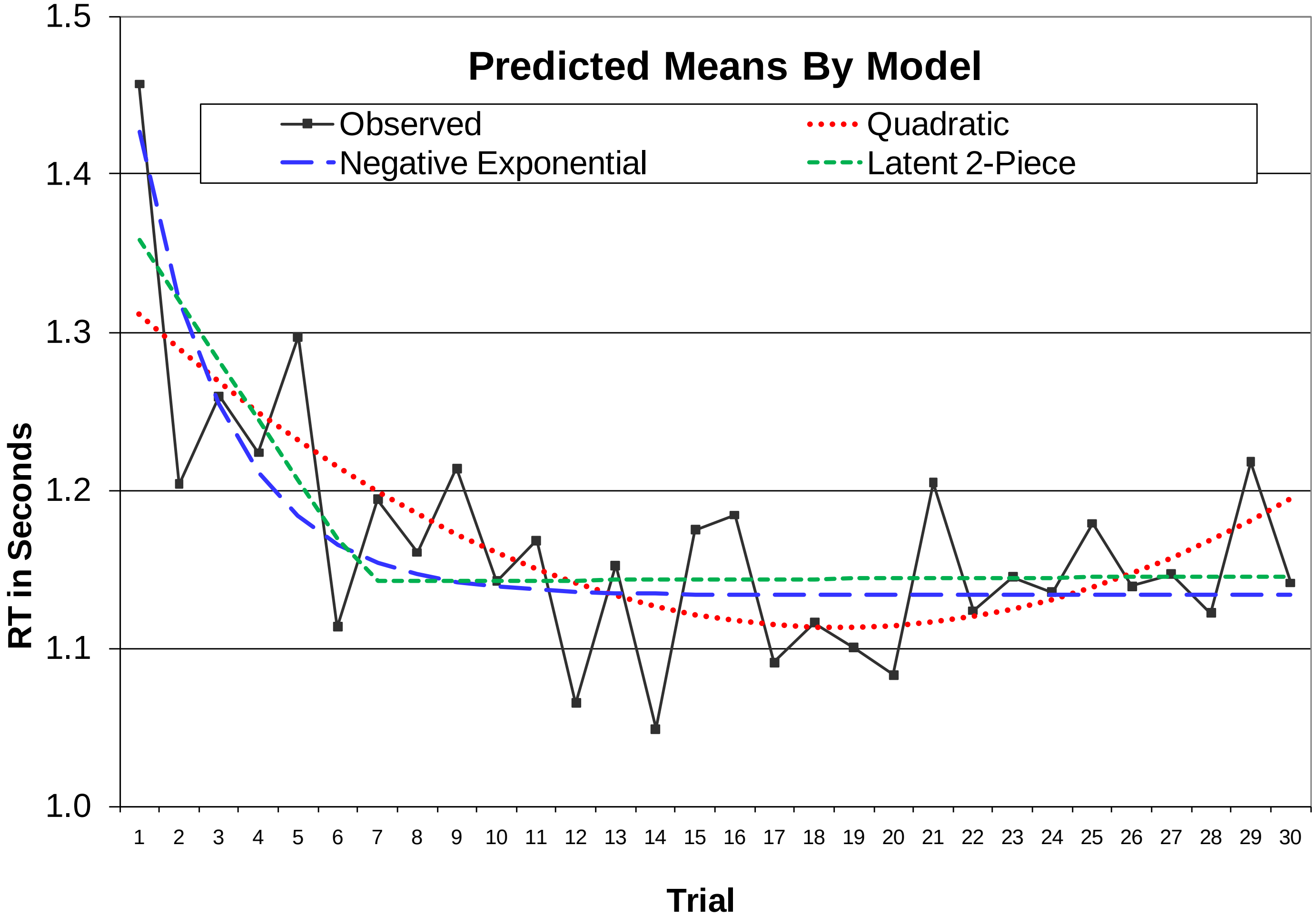
Session		1	2	3	4	5	6
Time	→ Slope16 =	0	1	2	3	4	5
Deviation	→ Slope26 =	0	0	1	2	3	4

Slope + Deviation Slope: Random Effects

- Parameters directly **differences across parts** of trajectory:
- **Fixed effects** for mean change over time (3 fixed effects):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope16 = expected linear rate of change from 1 to 2 (after controlling for slope26)
  - Fixed Slope26 = expected **extra** linear rate of change from 2 to 6 (after controlling for slope16, which is just time)
- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level when both slopes = 0 (at Session 1)
  - Random Slope16 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in **extra** linear slope from 2 to 6

Other Random Effects for Change

- **Truly nonlinear models**: Non-additive terms to describe change
  - Models can include **asymptotes** (so change can "shut off" as needed)
  - Include **power** and **exponential** functions (see chapter 6 for references)



Nonlinear Models

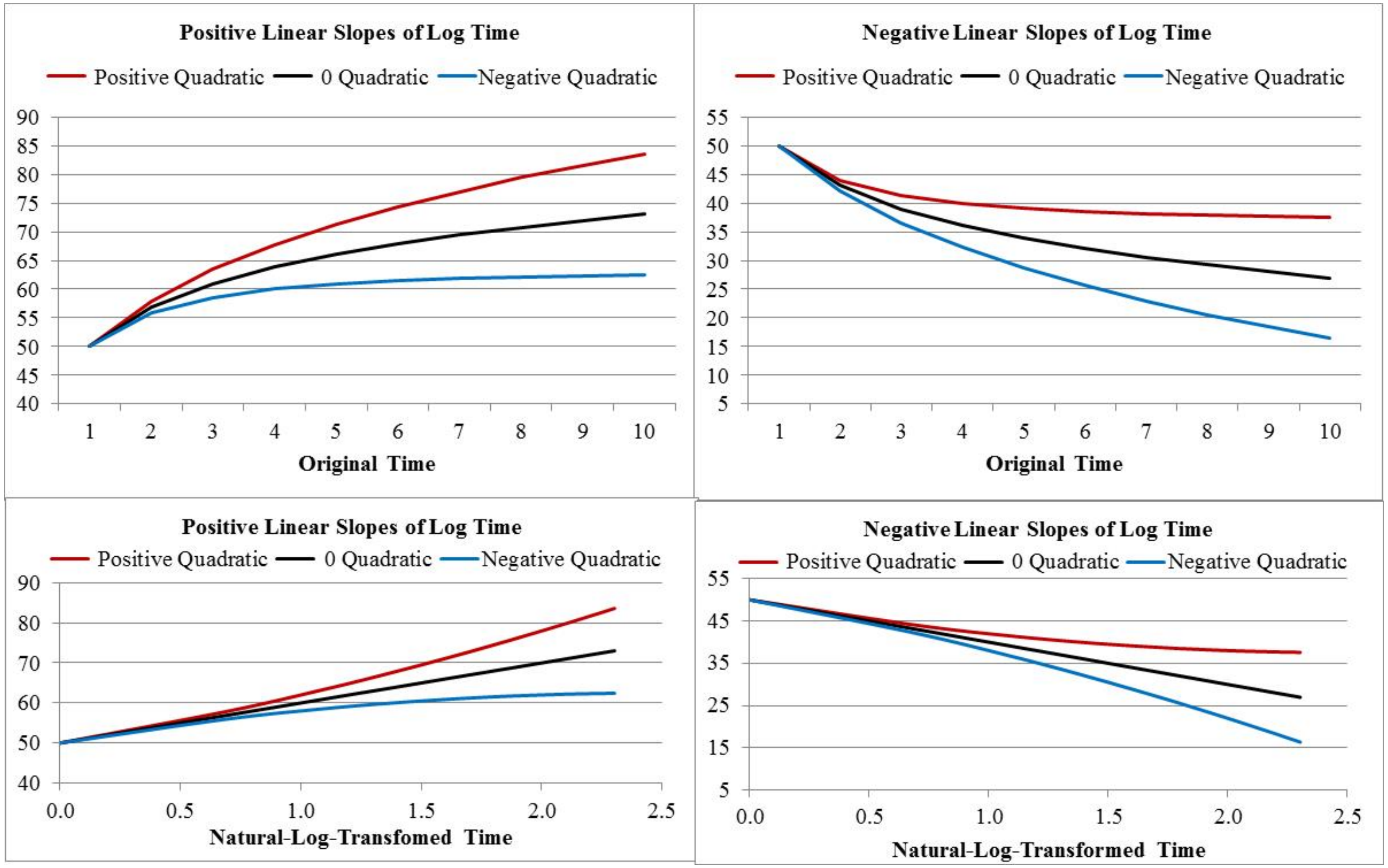
- Not all forms of change fit polynomial models
  - What goes up must come back down (and vice-versa)
  - Sometimes change needs to "shut off" (need asymptotes)

- Many kinds of truly nonlinear models can be used for longitudinal data
  - Linear in variables vs. linear in parameters (exp → nonlinear)
  - Logistic, power, exponential… see end of chapter 6 for ideas

- Require extra steps to evaluate estimation quality
  - Start values are needed, especially for random variances
  - Check that "gradient" values are as close to 0 as possible (partial first derivative of that parameter in LL function)

How to Mimic an Exponential Model

If you need to use REML, a predictor of natural-log-transformed time may be a good substitute for a truly nonlinear model



A <u>linear</u> effect of log time (black lines) predicts an <u>exponential</u> curve across *original* time.

Quadratic effects of log time (red or blue lines) can speed up or slow down the curve.

Bottom: There is a linear relationship between log-time and the outcome.

5. and 6. for **Systematic Change**: Evaluate Fixed and Random Effects of Time

Model for the Means:

- What kind of fixed effects of time are needed to create a function with which to parsimoniously represent the observed means across occasions?
  - Linear or nonlinear? Continuous or discontinuous?
  - Polynomials? Pieces? Log time? Truly nonlinear curves?
  - Use obtained *p*-values to test significance of fixed effects

Model for the Variance (focus primarily on G):

- What kind of random effects of time are needed:
  - To account for individual differences in aspects of change?
  - To describe the variances and covariances over time?
  - Do the residuals show any covariance after accounting for random effects?
  - Use REML −2ΔLL tests to test significance of new effects (or ML if big *N*)

Random Effects Models for the Variance

- Each source of correlation or dependency goes into a new variance component (or pile of variance) until each source meets the usual assumptions of GLM: normality, independence, constant variance
- **Example 2-level longitudinal model:**

Level 1 (one source of) <u>Within-Person Variation</u>: gets accounted for by time-level predictors

Residual Variance (σ_e^2)

FIXED effects make variance go away (explain variance).

RANDOM effects just make a new pile of variance.

Level 2 (two sources of) <u>Between-Person Variation</u>: gets accounted for by person-level predictors

BP Int Variance ($\tau_{U_0}^2$)

BP Slope Variance ($\tau_{U_1}^2$)

$\tau_{U_{01}}$ **covariance**

Now we get to add predictors to account for each pile!

5. for **NO Systematic Change**:
Evaluate Alternative Covariance Structures

Model for the Means:

- Be sure you don't need any terms for systematic effects of time
- If not, keep a fixed intercept only

Model for the Variance (focus primarily on R):

- How many parameters do you need to predict the original data?
- I recommend the hybrid: Random Intercept in **G** + Structure in **R**
  - Separates BP and WP variance
  - Likely more parsimonious than just **R**-only model
- Compare alternative models with the same fixed effects
  - Nested? REML −2ΔLL test for significance
  - Non-nested? REML AIC and BIC for "supporting evidence"

Alternative Covariance Structure Models

- Models for fluctuation typically include only a covariance structure, and at most a random intercept (random slopes for time won't help in the absence of systematic change)

Between-Person Random Intercept in G + Within-Person Structure in R

Level 1 (one source of) Within-Person Variation:

Gets accounted for by time-level predictors

Residual Variance (σ_e^2)

Level 2 (one sources of) Between-Person Variation:

Gets accounted for by person-level predictors

BP Int Variance ($\tau_{U_0}^2$)

TOTAL Structure in R

All sources of variation and covariation are held in one matrix, but if dependency is predicted accurately then it's ok.

Total Variance (σ_T^2)

Why spend so much effort on unconditional models of time? Here is the reasoning…

- The fixed effects of time are what the random effects of time are varying around…
- The random effects of time form the variances that the person-level predictors will account for…
- The effects of person-level predictors are specified as a function of the time effects already in the model…
- The effects of time-varying predictors are supposed to account for variance not accounted for by the model for time…
- What fixed and random time effects of time you include in the model dictate what is to be predicted. **Get time right first!**