Example 6: Explanatory IRT Models as Crossed Random Effects Models in SAS GLIMMIX and STATA MELOGIT using Laplace Maximum Likelihood Estimation

This example shows variants of "explanatory" item response theory (IRT) models, which are actually generalized multilevel models with random subject intercepts and either fixed or random effects for items. These example data are from my dissertation: 36 items assessing attentional search via change detection scored incorrect (cor=0) or correct (cor=1). Items varied by four dimensions: continuous visual clutter (Rclut), whether the change was relevant to driving (Rrel), continuous brightness of the change (Rbrit), and whether the change was made to a legible sign (LegSign). These analyses require a "stacked" (or "long") format in which each item for each person is stored on a separate row.

SAS Code to read in already-stacked data:

```
* Reading in data and keeping items with item predictor variables;
DATA work.o38stack; SET diss.work.o38stack; WHERE NMISS(cor, rclut, rrel, rbrit, legsign)=0; RUN;
PROC SORT DATA=work.o38stack; BY PartID picture; RUN;
```

STATA Code to read in already-stacked data:

```
// Import example stata data file
use "$filesave\o38stack.dta", clear

// Keep items with item predictor variables
egen nmiss=rowmiss(cor rclut rrel rbrit legsign)
drop if nmiss>0
drop if use38==0
```

Single-Level Empty Model with no Random Effects: Logit(y_{tpi} =1) = γ_{000}

```
TITLE "SAS Single-Level Empty Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) =  / SOLUTION LINK=LOGIT DIST=BINARY;  * Binary response, logit link;
     ESTIMATE "Intercept" intercept 1 / ILINK; * Inverse: Logit to probability;
RUN; TITLE;
```

```
display as result "STATA Single-Level Empty Model"
melogit cor ,  intmethod(laplace),
nlcom 1/(1+exp(-1*(_b[_cons]))) // fixed intercept in probability
```

```
          Fit Statistics
-2 Log Likelihood            5685.63
AIC  (smaller is better)     5687.63
AICC (smaller is better)     5687.63
BIC  (smaller is better)     5694.23
CAIC (smaller is better)     5695.23
HQIC (smaller is better)     5689.94
Pearson Chi-Square           5426.00
Pearson Chi-Square / DF         1.00

                          Parameter Estimates
                      Standard
Effect       Estimate       Error       DF    t Value    Pr > |t|    Gradient
Intercept      1.2794     0.03290     5425      38.89      <.0001    -992E-12 → mean easiness in logits

                                   Estimates
                                                                        Standard
                      Standard                                             Error
Label        Estimate       Error       DF    t Value    Pr > |t|        Mean        Mean
Intercept      1.2794     0.03290     5425      38.89      <.0001      0.7823    0.005602 → mean probability
```

Empty Model with Random Subjects Only: Logit(y_{tpi} =1) = γ_{000} + U_{0p0}

```
TITLE "SAS Random Subjects, Empty Items Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) =  / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=PartID; * theta/U0 per subject;
     ESTIMATE "Intercept" intercept 1 / ILINK; * Inverse: Logit to probability;
     COVTEST "Need subject random int?" 0;
RUN; TITLE;

display as result "STATA Random Subjects, Empty Items Model"
melogit cor , || partid: , covariance(unstructured) intmethod(laplace),
nlcom 1/(1+exp(-1*(_b[_cons]))) // fixed intercept in probability
```

```
          Fit Statistics
-2 Log Likelihood            5600.45
AIC  (smaller is better)     5604.45
AICC (smaller is better)     5604.45
BIC  (smaller is better)     5610.54
CAIC (smaller is better)     5612.54
HQIC (smaller is better)     5606.92

    Fit Statistics for Conditional
              Distribution
-2 log L(COR | r. effects)     5373.21
Pearson Chi-Square             5107.44
Pearson Chi-Square / DF           0.94

      Covariance Parameter Estimates
Cov                                Standard
Parm       Subject    Estimate       Error    Gradient
UN(1,1)    PARTID       0.2524     0.05033    -0.00018 → variance of theta (mean = 0): persons are random

                  Solutions for Fixed Effects
                        Standard
Effect       Estimate       Error       DF    t Value    Pr > |t|    Gradient
Intercept      1.3487     0.05367      154      25.13      <.0001    0.000155 → item mean easiness in logits

                                     Estimates
                                                                          Standard
                        Standard                                             Error
Label        Estimate       Error       DF    t Value    Pr > |t|        Mean        Mean
Intercept      1.3487     0.05367      154      25.13      <.0001      0.7939    0.008781 →unit-specific int

                        Tests of Covariance Parameters
                           Based on the Likelihood
Label                         DF    -2 Log Like      ChiSq    Pr > ChiSq    Note
Need subject random int?       1        5685.63      85.18        <.0001    MI → correct!
MI: P-value based on a mixture of chi-squares.
```

Explanatory Items Model with <u>Random Subjects Only</u> (LLTM-Predicted Item Easiness):

Logit(y_{tpi} =1) = γ_{000} + γ_{001}($Rclut_i$) + γ_{002}($Rrel_i$) + γ_{003}($Rbrit_i$) + γ_{004}($LegSign_i$) + U_{0p0}

```
TITLE "SAS Random Subjects, LLTM-Predicted Fixed Items Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) = rclut rrel rbrit legsign / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=PartID;
RUN; TITLE;
```

```
display as result "STATA Random Subjects, LLTM-Predicted Fixed Items Model"
melogit cor rclut rrel rbrit legsign, || partid: , covariance(unstructured) intmethod(laplace)
```

```
           Fit Statistics
-2 Log Likelihood            5439.80
AIC  (smaller is better)     5451.80
AICC (smaller is better)     5451.81
BIC  (smaller is better)     5470.06
CAIC (smaller is better)     5476.06
HQIC (smaller is better)     5459.21

    Fit Statistics for Conditional
              Distribution
-2 log L(COR | r. effects)     5206.49
Pearson Chi-Square             5125.58
Pearson Chi-Square / DF           0.94

      Covariance Parameter Estimates
Cov                                Standard
Parm       Subject    Estimate       Error    Gradient
UN(1,1)    PARTID       0.2734     0.05360    -0.0000 → theta variance is bigger now due to L1 predictors

                  Solutions for Fixed Effects
                        Standard
Effect       Estimate       Error       DF    t Value    Pr > |t|    Gradient
Intercept      0.8619      0.1529      154       5.64      <.0001    -0.00104 → mean logit easiness if x=0
RCLUT         -0.2675     0.05548     5267      -4.82      <.0001    -0.00001 → Δ in logit/unit clutter
RREL           0.2204     0.09936     5267       2.22      0.0266    -0.00149 → Δ in logit if relevant
RBRIT          0.4742      0.1129     5267       4.20      <.0001    -0.00061 → Δ in logit/unit brightness
LEGSIGN        0.6621     0.08223     5267       8.05      <.0001    9.234E-6 → Δ in logit if legible sign
```

Rasch Model with Random Subjects and Saturated Fixed Items (via a categorical item predictor):

$$\text{Logit}(y_{tpi}=1) = \gamma_{000} + \gamma_{001}(\text{Pic2}_i) + \gamma_{002}(\text{Pic3}_i) + \ldots. + \gamma_{0035}(\text{Pic35}_i) + U_{0p0}$$

```
TITLE "SAS Random Subjects, Rasch Saturated Fixed Items Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) = picture / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=PartID;
     LSMEANS picture; * Get logit intercept per item;
     ODS OUTPUT LSMEANS=Rasch; * Save to dataset;
RUN; TITLE;
```

```
display as result "STATA Random Subjects, Rasch Saturated Fixed Items Model"
melogit cor i.picture, || partid: , covariance(unstructured) intmethod(laplace)
        margins i.picture, predict(xb)  // Get logit intercept per item
```

```
           Fit Statistics
-2 Log Likelihood            4907.49
AIC  (smaller is better)     4981.49
AICC (smaller is better)     4982.01
BIC  (smaller is better)     5094.09
CAIC (smaller is better)     5131.09
HQIC (smaller is better)     5027.22

    Fit Statistics for Conditional
              Distribution
-2 log L(COR | r. effects)     4650.39
Pearson Chi-Square             4959.17
Pearson Chi-Square / DF           0.91
```

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Gradient
UN(1,1)	PARTID	0.3678	0.06794	-0.00004 → theta variance is bigger now due to L1 predictors

Type III Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
PICTURE	35	5236	15.41	<.0001 → items do vary significantly from each other in easiness

PICTURE Least Squares Means → easiness fixed effect estimates per item (saved to dataset)

Picture Number	Estimate	Standard Error	DF	t Value	Pr > |t|
2	0.2306	0.1781	5236	1.29	0.1957
6	3.3655	0.4235	5236	7.95	<.0001
7	1.1464	0.1989	5236	5.76	<.0001
10	2.2291	0.2673	5236	8.34	<.0001
11	0.9651	0.1920	5236	5.03	<.0001
13	0.1150	0.1764	5236	0.65	0.5145
22	1.2780	0.2060	5236	6.21	<.0001
23	2.9246	0.3515	5236	8.32	<.0001
26	1.0325	0.1947	5236	5.30	<.0001
33	1.6398	0.2243	5236	7.31	<.0001
35	1.3106	0.2078	5236	6.31	<.0001
42	0.9466	0.1929	5236	4.91	<.0001
52	2.7153	0.3219	5236	8.44	<.0001
59	2.2571	0.2746	5236	8.22	<.0001
61	0.8884	0.1927	5236	4.61	<.0001
62	-0.3283	0.1801	5236	-1.82	0.0684
66	2.5329	0.2993	5236	8.46	<.0001
97	1.3749	0.2090	5236	6.58	<.0001
117	3.0615	0.3706	5236	8.26	<.0001
123	2.3657	0.2817	5236	8.40	<.0001
128	1.1477	0.1989	5236	5.77	<.0001
135	3.3718	0.4235	5236	7.96	<.0001
136	1.2737	0.2092	5236	6.09	<.0001
137	1.4945	0.2161	5236	6.92	<.0001
140	0.3818	0.1809	5236	2.11	0.0348
146	1.7911	0.2338	5236	7.66	<.0001
152	0.7385	0.1874	5236	3.94	<.0001
155	-0.2088	0.1783	5236	-1.17	0.2416
161	2.9072	0.3519	5236	8.26	<.0001
162	3.0362	0.3709	5236	8.19	<.0001
171	1.5756	0.2210	5236	7.13	<.0001
172	0.8620	0.1910	5236	4.51	<.0001
173	3.2169	0.3938	5236	8.17	<.0001
174	1.6938	0.2272	5236	7.45	<.0001
177	0.8978	0.1897	5236	4.73	<.0001
179	1.7847	0.2340	5236	7.63	<.0001

Empty Model with Random Subjects AND Random Items: Logit(y_{tpi} =1) = γ_{000} + U_{0p0} + U_{00i}

```
TITLE "SAS Random Subjects, Random Items Empty Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) =  / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=PartID;  * Each subject gets a theta/U0;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=picture; * Each item gets an easiness;
     COVTEST "Need item random int?" . 0;
     COVTEST "Need subject random int?" 0 .;
     ODS OUTPUT SolutionR=Crossed;
RUN; TITLE;

* Creating analogous individual item effects to fixed effects items model;
DATA work.Crossed; SET work.Crossed; WHERE INDEX(Subject, "PICTURE")>0;
     Random_b=1.5928 + Estimate; * Add fixed intercept back into each item random effect;
     KEEP Random_b; RUN;

display as result "STATA Random Subjects, Random Items Empty Model"
melogit cor , || _all: R.picture || partid: , covariance(unstructured) intmethod(laplace)
```

Fit Statistics	
-2 Log Likelihood	5049.84
AIC (smaller is better)	5055.84
AICC (smaller is better)	5055.84
BIC (smaller is better)	5049.84
CAIC (smaller is better)	5052.84
HQIC (smaller is better)	5049.84

Fit Statistics for Conditional Distribution	
-2 log L(COR | r. effects)	4656.19
Pearson Chi-Square	4735.50
Pearson Chi-Square / DF	0.87

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Gradient	
UN(1,1)	PARTID	0.3625	0.06733	0.004887	→ theta variance
UN(1,1)	PICTURE	0.9492	0.2431	-0.00131	→ item easiness variance

Solutions for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr > |t|	Gradient	
Intercept	1.5928	0.1750	35	-9.10	<.0001	0.000010	→ mean item easiness in logits

Solution for Random Effects → deviation of each item from 'mean' easiness (saved to dataset)

Effect	Subject	Estimate	Std Err Pred	DF	t Value	Pr > |t|
Intercept	PICTURE 2	1.3266	0.2337	5236	5.68	<.0001
Intercept	PICTURE 6	-1.4697	0.3819	5236	-3.85	0.0001
Intercept	PICTURE 7	0.4583	0.2475	5236	1.85	0.0641
Intercept	PICTURE 10	-0.5488	0.2935	5236	-1.87	0.0616
Intercept	PICTURE 11	0.6304	0.2430	5236	2.59	0.0095
Intercept	PICTURE 13	1.4364	0.2326	5236	6.18	<.0001
Intercept	PICTURE 22	0.3336	0.2524	5236	1.32	0.1863
Intercept	PICTURE 23	-1.1401	0.3452	5236	-3.30	0.0010
Intercept	PICTURE 26	0.5663	0.2448	5236	2.31	0.0207
Intercept	PICTURE 33	-0.00675	0.2648	5236	-0.03	0.9797
Intercept	PICTURE 35	0.3027	0.2537	5236	1.19	0.2329
Intercept	PICTURE 42	0.6476	0.2436	5236	2.66	0.0079
Intercept	PICTURE 52	-0.9695	0.3278	5236	-2.96	0.0031
Intercept	PICTURE 59	-0.5725	0.2985	5236	-1.92	0.0552

Intercept	PICTURE 61	0.7024	0.2434	5236	2.89	0.0039
Intercept	PICTURE 62	1.8513	0.2350	5236	7.88	<.0001
Intercept	PICTURE 66	-0.8153	0.3140	5236	-2.60	0.0094
Intercept	PICTURE 97	0.2422	0.2545	5236	0.95	0.3413
Intercept	PICTURE 117	-1.2488	0.3555	5236	-3.51	0.0004
Intercept	PICTURE 123	-0.6698	0.3029	5236	-2.21	0.0270
Intercept	PICTURE 128	0.4571	0.2475	5236	1.85	0.0648
Intercept	PICTURE 135	-1.4751	0.3818	5236	-3.86	0.0001
Intercept	PICTURE 136	0.3371	0.2546	5236	1.32	0.1856
Intercept	PICTURE 137	0.1295	0.2593	5236	0.50	0.6175
Intercept	PICTURE 140	1.1831	0.2356	5236	5.02	<.0001
Intercept	PICTURE 146	-0.1478	0.2713	5236	-0.54	0.5859
Intercept	PICTURE 152	0.8449	0.2398	5236	3.52	0.0004
Intercept	PICTURE 155	1.7400	0.2339	5236	7.44	<.0001
Intercept	PICTURE 161	-1.1244	0.3456	5236	-3.25	0.0011
Intercept	PICTURE 162	-1.2262	0.3560	5236	-3.44	0.0006
Intercept	PICTURE 171	0.05319	0.2627	5236	0.20	0.8395
Intercept	PICTURE 172	0.7276	0.2423	5236	3.00	0.0027
Intercept	PICTURE 173	-1.3674	0.3675	5236	-3.72	0.0002
Intercept	PICTURE 174	-0.05721	0.2668	5236	-0.21	0.8302
Intercept	PICTURE 177	0.6943	0.2414	5236	2.88	0.0040
Intercept	PICTURE 179	-0.1423	0.2714	5236	-0.52	0.6002

Tests of Covariance Parameters
Based on the Likelihood

Label	DF	-2 Log Like	ChiSq	Pr > ChiSq	Note
Need item random int?	1	5600.45	550.61	<.0001	MI → correct
Need subject random int?	1	.	.	.	→ not sure what happened?

MI: P-value based on a mixture of chi-squares.

Both items and subjects need to have random intercepts per my own LRTs…

Explanatory Items Model with Random Subjects AND Random Items (Predicted Item Easiness)

$$\text{Logit}(y_{tpi}=1) = \gamma_{000} + \gamma_{001}(\text{Rclut}_i) + \gamma_{002}(\text{Rrel}_i) + \gamma_{003}(\text{Rbrit}_i) + \gamma_{004}(\text{LegSign}_i) + U_{0p0} + U_{00i}$$

```
TITLE "SAS Random Subjects, LLTM + Random Items Model";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) = rclut rrel rbrit legsign / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=PartID;  * Each subject gets a theta/U0;
     RANDOM INTERCEPT / SOLUTION TYPE=UN SUBJECT=picture; * Each item gets an easiness leftover;
RUN; TITLE;

display as result "STATA Random Subjects, LLTM + Random Items Model"
melogit cor rclut rrel rbrit legsign, || _all: R.picture || partid: , ///
        covariance(unstructured) intmethod(laplace)
```

Fit Statistics	
-2 Log Likelihood	5040.20
AIC (smaller is better)	5054.20
AICC (smaller is better)	5054.22
BIC (smaller is better)	5040.20
CAIC (smaller is better)	5047.20
HQIC (smaller is better)	5040.20

Fit Statistics for Conditional Distribution	
-2 log L(COR | r. effects)	4656.19
Pearson Chi-Square	4735.50
Pearson Chi-Square / DF	0.87

```
           Covariance Parameter Estimates
Cov                              Standard
Parm       Subject    Estimate       Error    Gradient
UN(1,1)    PARTID       0.3625     0.06732    0.002419 → theta variance across subjects
UN(1,1)    PICTURE      0.7138      0.1858    0.000701 → remaining variance in item easiness (down 24.8%)

                        Solutions for Fixed Effects
                        Standard
Effect       Estimate       Error       DF    t Value    Pr > |t|    Gradient
Intercept      1.3105      0.6354       32       2.06      0.0474    0.002994 → mean easiness when x=0
RCLUT         -0.3239      0.2420     5235      -1.34      0.1809    0.000473 → Δ in logit/unit clutter
RREL         0.03715      0.4261     5235       0.09      0.9305    0.004363 → Δ in logit if relevant
RBRIT         0.7896      0.4989     5235       1.58      0.1136    4.913E-6 → Δ in logit/unit brightness
LEGSIGN       0.7388      0.3367     5235       2.19      0.0283    0.000287 → Δ in logit if legible sign
```

Comparing Results: Fixed vs. Random Effects for Item Easiness Predictions

```
DATA work.items; SET work.o38stack; WHERE PartID=201; RUN;
DATA work.Merged; MERGE work.Rasch (RENAME=(Estimate=Fixed_b)) work.Crossed work.items; RUN;
DATA work.Merged; SET Merged;
LLTM_pred_b    = 0.8619 + (-.2675*rclut) + (.22040*rrel) + (.4742*rbrit) + (.6621*legsign);
Crossed_pred_b = 1.3105 + (-.3239*rclut) + (.03715*rrel) + (.7896*rbrit) + (.7388*legsign);
RUN;
PROC CORR DATA=work.Merged; VAR fixed_b random_b LLTM_pred_b Crossed_pred_b; RUN;
                                  Simple Statistics
Variable                 N          Mean       Std Dev           Sum       Minimum       Maximum
Fixed_b                 36       1.61155       1.02674      58.01583      -0.32828       3.37183
Random_b                36       1.63255       0.93089      58.77182       0.11072       3.43713
LLTM_pred_b             36       1.40093       0.44574      50.43335       0.60327       2.28516
Crossed_pred_b          36       1.59253       0.49180      57.33107       0.53570       2.58009

Pearson Correlation Coefficients, N = 36; Prob > |r| under HO: Rho=0
                                                    LLTM_      Crossed_
                        Fixed_b      Random_b        pred_b        pred_b
Fixed_b                 1.00000       0.99950       0.47628       0.48551
                                       <.0001        0.0033        0.0027
Random_b                0.99950       1.00000       0.47770       0.48561
                         <.0001                      0.0032        0.0027
LLTM_pred_b             0.47628       0.47770       1.00000       0.98472
                          0.0033        0.0032                      <.0001
Crossed_pred_b          0.48551       0.48561       0.98472       1.00000
                          0.0027        0.0027        <.0001
```

The R^2 for item easiness is around .24, which is very close to what we found as the proportion reduction in item variance after including the 4 item predictors.

The difference lies in the significance of the item predictor effects, which are based on the wrong df and error term in the random subjects only LLTM.

Explanatory Items Model with Random Subjects AND Random Items (Predicted Item Easiness): Example of Adding a Random Slope of an Item Predictor over Subjects

$$\text{Logit}(y_{tpi}=1) = \gamma_{000} + \gamma_{001}(\text{Rclut}_i) + \gamma_{002}(\text{Rrel}_i) + \gamma_{003}(\text{Rbrit}_i) + \gamma_{004}(\text{LegSign}_i) + U_{0p0} + U_{0p3}(\text{Rbrit}_i) + U_{00i}$$

```
TITLE1 "SAS Add Random Brightness Slope (NS)";
PROC GLIMMIX DATA=work.o38stack NOCLPRINT NOITPRINT GRADIENT METHOD=Laplace;
     CLASS PartID picture;
     MODEL cor (DESCENDING) = rclut rrel rbrit legsign / SOLUTION LINK=LOGIT DIST=BINARY;
     RANDOM INTERCEPT rbrit / TYPE=UN SUBJECT=PartID;  * Each subject gets a theta;
     RANDOM INTERCEPT / TYPE=UN SUBJECT=picture; * Each item gets an easiness;
     COVTEST "Need Random rbrit Slope?" . 0 0 ;
RUN;
                      Tests of Covariance Parameters
                         Based on the Likelihood
Label                       DF    -2 Log Like      ChiSq    Pr > ChiSq    Note
Need Random rbrit Slope?     2              .          .             .            → should be -2LL diff=1.59
```