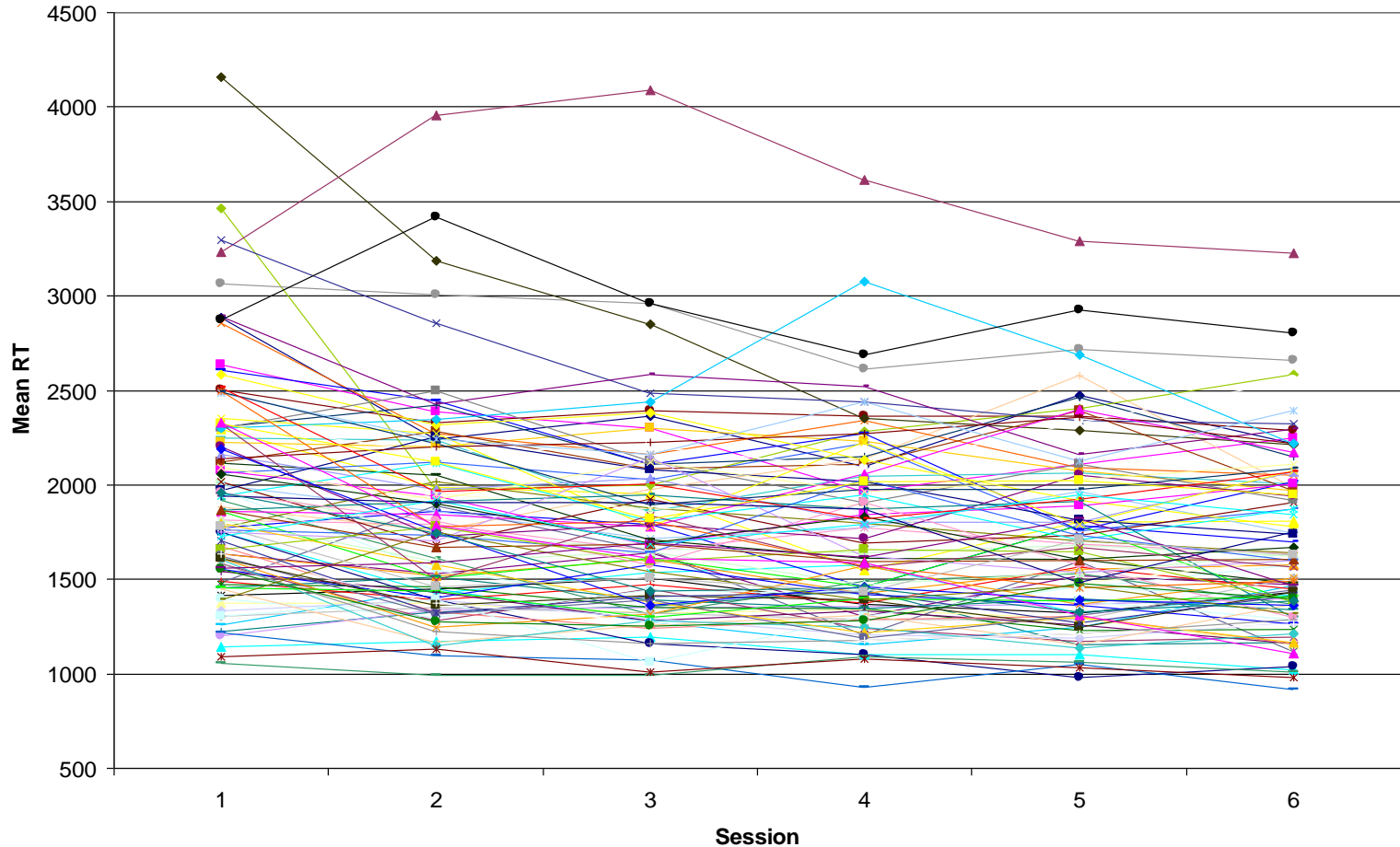


Describing Within-Person Change over Time

- Topics:
 - The big picture of modeling change
 - Fixed and random effects models for nonlinear change:
 - Polynomial slopes
 - Piecewise slopes
 - Truly nonlinear change (via exponential curves and linear approximations thereof)
 - Bonus: translation of change models into SEM

Example Data Individual Observed Trajectories ($N = 101, n = 6$)

Number Match 3 Response Times by Session

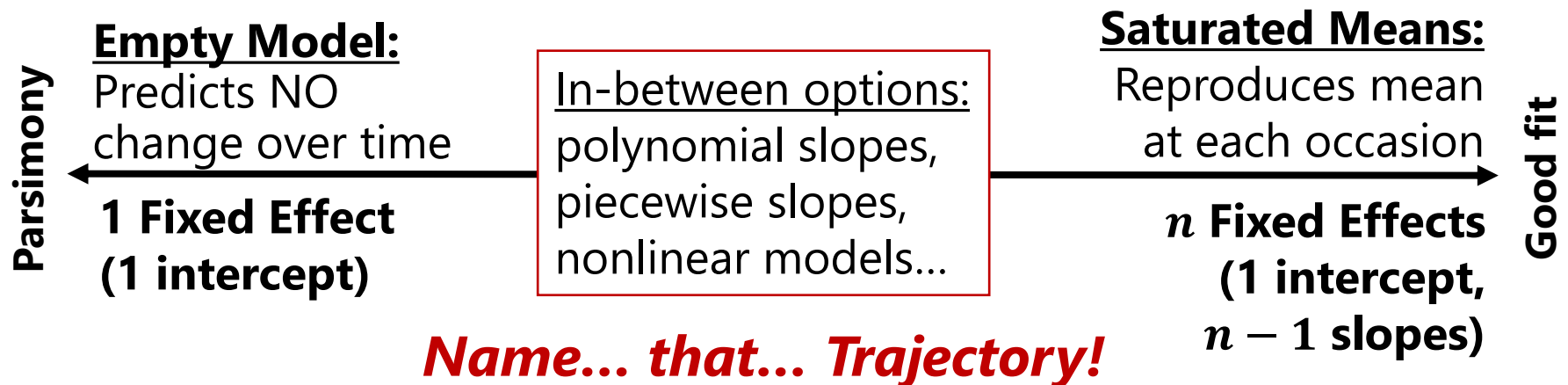


The Big Picture of Longitudinal Data: Model for the Means (Fixed Effects)

- What kind of change occurs on average over “time”?
 - What is the most appropriate **metric of time**?
 - Time in study (with other predictors for BP differences in time)?
 - Time since birth (age)? Time to event (time since diagnosis)?
 - Measurement occasions need not be the same across persons or equally spaced (so represent time as exactly as possible)
 - What kind of **theoretical process** generated the observed trajectories, and thus what kind of model do we need?
 - Linear or nonlinear? Continuous or discontinuous? Does change keep happening or should it eventually stop?
 - Many options: polynomial, piecewise, and nonlinear families

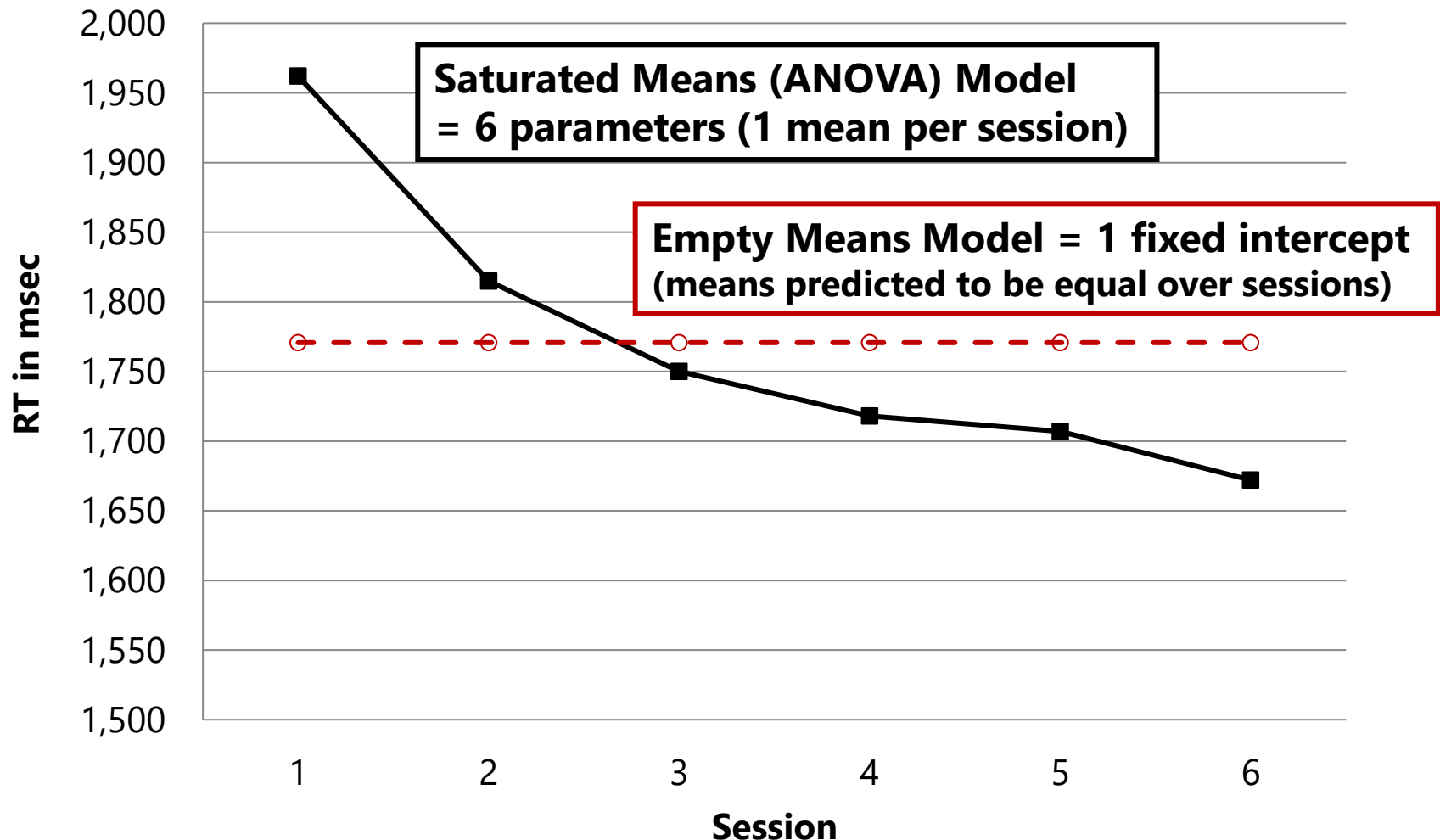
The Big Picture of Longitudinal Data: Alternative Models for the Means

- What kind of change occurs on average over time? Here are two baseline models:
 - **"Empty"** → only a fixed intercept (predicts no change)
 - **"Saturated"** → all occasion mean differences from time 0 (ANOVA model that uses n fixed effects)
**** may not be possible in unbalanced data*



Baseline Models for the Means

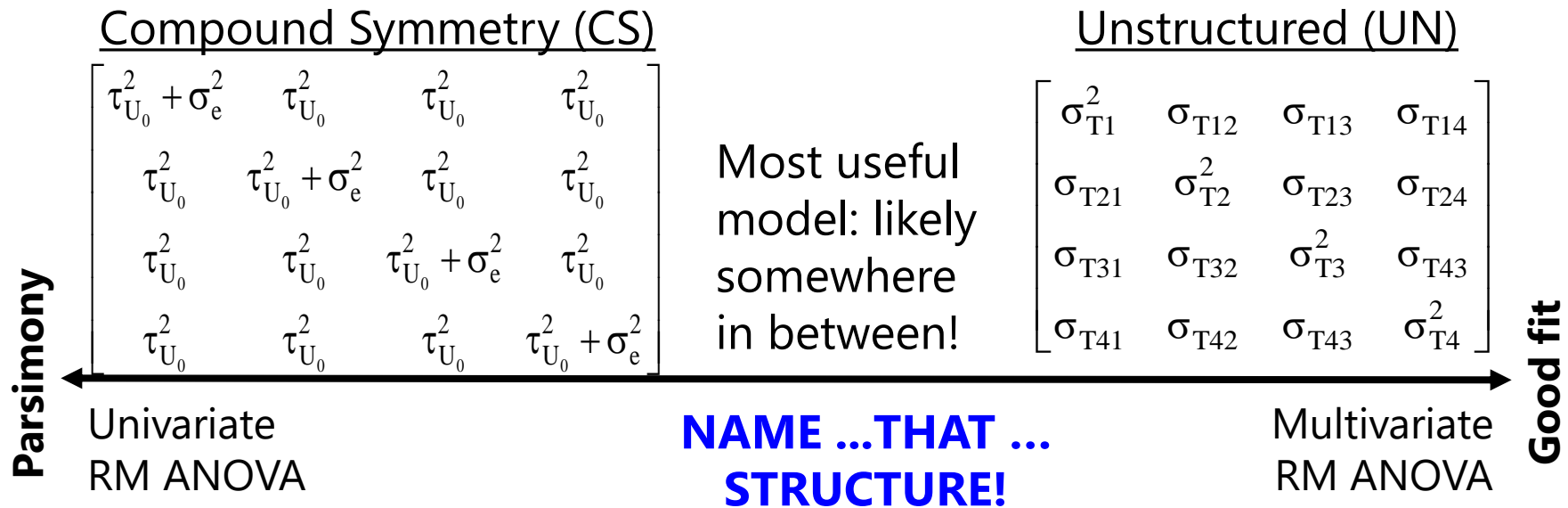
Number Match 3 Mean Response Times by Session



The Big Picture of Longitudinal Data: Models for the Variance (Random Effects)

- *From a substantive perspective:*
Are there **individual differences** in change?
 - Individual differences in the intercept of an outcome?
 - At what occasion are individual differences in outcome level important for your hypotheses (beginning, middle, or end)?
 - Individual differences in the magnitude of change?
 - Each aspect of change (e.g., linear change, quadratic change) can potentially exhibit individual differences (data permitting)
- *From a statistical perspective:* What kind of pattern do the **variances and covariances** exhibit over time?
 - Do the variances increase or decrease over time?
 - Are the covariances differentially related based on time?

The Big Picture of Longitudinal Data: Alternative Models for the Variance

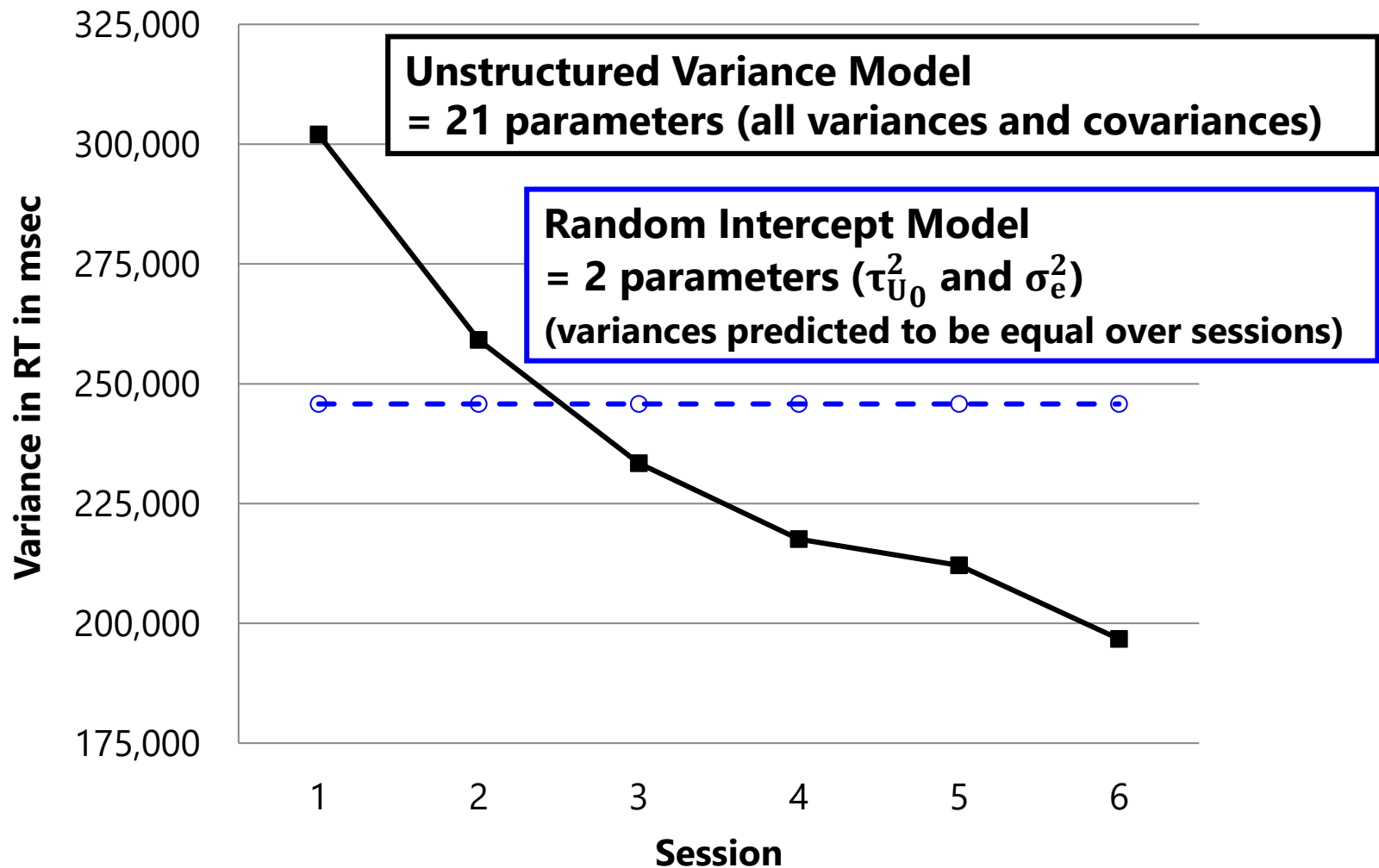


What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Baseline Models for the Variance

Variance in Number Match 3 Response Times by Session



Summary: Modeling **Means** and **Variance**

- We have two tasks in describing within-person change:
- **Choose a Model for the Means**
 - What kind of change in the outcome do we have on average?
 - What kind and how many **fixed effects** do we need to predict the mean trajectory as parsimoniously but accurately as possible?
- **Choose a Model for the Variance**
 - What pattern do the variances and covariances of the outcome show over time because of individual differences in change?
 - What kind and how many **random effects** do we need to predict that pattern as parsimoniously but accurately as possible?

Bonus: Testing Absolute Fit in REML

- **Answer key model** (only possible directly for balanced data):
 - **Model for the Means** = Saturated Means (fixed effects = n occasions)
 - **Model for the Variance** = Unstructured R, or $RI+UN(n-1)$ equivalent
- Tests of absolute fit of any **simpler means model** against saturated means can only be done via $-2\Delta LL$ when using ML, but what if you need to **use REML** given small level-2 N ?
 - Use a **multivariate Wald test** instead: add enough contrasts for occasion-specific mean differences to create saturated means, then test that group of contrasts (see Example 6)
 - Idea from this book: [Generalized Linear Mixed Models \(Stroup et al.\)](#)
- Tests of absolute fit of any **simpler variance model against UN** can be done using REML $-2\Delta LL$ given same model for the means (so keep the same fixed time slopes in each model)

Name that trajectory... Polynomial?

- Predict **mean change** with **polynomial fixed effects of time**:
 - Linear → *constant* amount of change (up or down)
 - Quadratic → *change* in linear rate of change (acceleration/deceleration)
 - Cubic → *change* in acceleration/deceleration of linear rate of change (known in physics as jerk, surge, or jolt)
 - Terms work together to describe curved trajectories (most useful for continuous trends that change direction)
- **Can have polynomial fixed time slopes UP TO: $n - 1$ ***
 - 3 occasions = 2nd order (time²) = Fixed Quadratic Time or less
 - 4 occasions = 3rd order (time³) = Fixed Cubic Time or less
- Interpretable polynomials past cubic are rarely seen in practice

**This rule can be broken in unbalanced data (but cautiously)*

Interpreting Quadratic Fixed Effects

A Quadratic time slope is a two-way interaction: time*time

- Fixed quadratic time = “**half the rate of acceleration/deceleration**”
- So to interpret it as how the linear time slope changes *per unit time*, **you must multiply the quadratic slope coefficient by 2**
- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
 - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6...
- Btw, the “twice” part comes from taking the derivatives of the function:

$$\text{Intercept (position) at } x_i = x: y_x = 50 + 4(x_i) + 0.3(x_i^2)$$

$$\text{First derivative (velocity) at } x: \frac{dy_x}{d(x)} = 4 + (2 * 0.3)(x_i)$$

$$\text{Second derivative (acceleration) at } x: \frac{d^2y_x}{d(x)} = (2 * 0.3)$$

Interpreting Quadratic Fixed Effects

A Quadratic time effect is a two-way interaction: time*time

- Fixed quadratic = “**half the rate of acceleration/deceleration**”
- So to interpret it as how the linear time slope changes *per unit time*, **you must multiply the quadratic slope coefficient by 2**
- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
 - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6...

- The “twice” part also comes from what you remember about the role of interactions with respect to their constituent main effects:

$$\hat{y}_i = \beta_0 + \beta_1(x_i) + \beta_2(z_i) + \beta_3(x_i)(z_i)$$

$$\text{Slope of } x_i: \beta_1 + \beta_3(z_i)$$

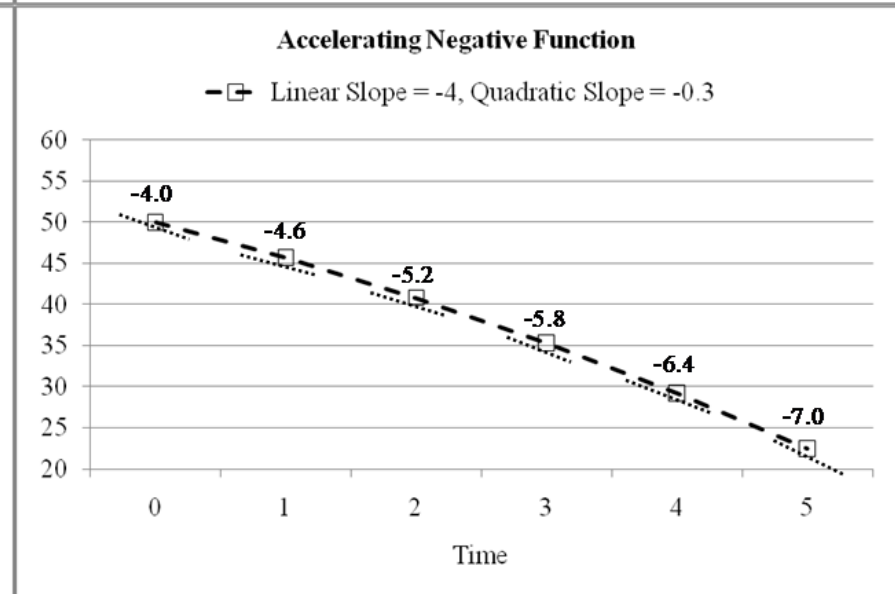
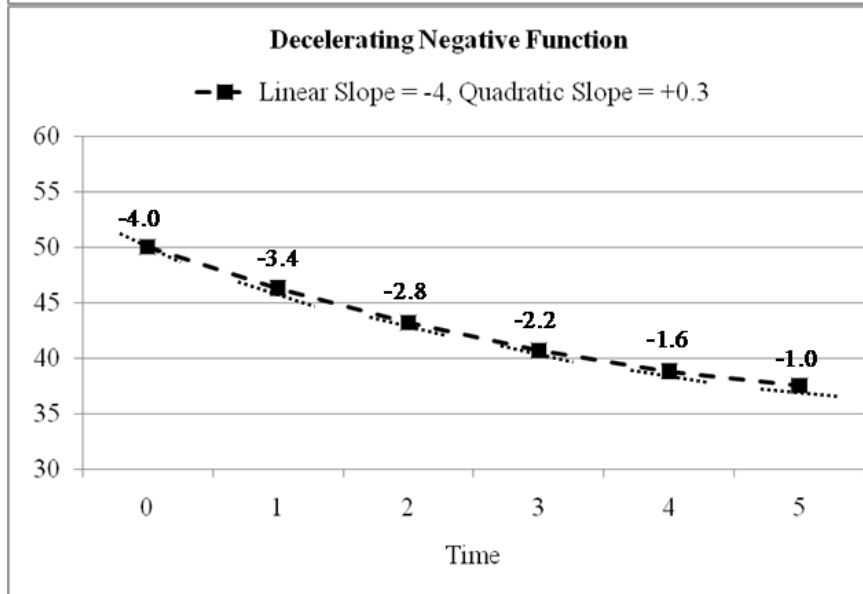
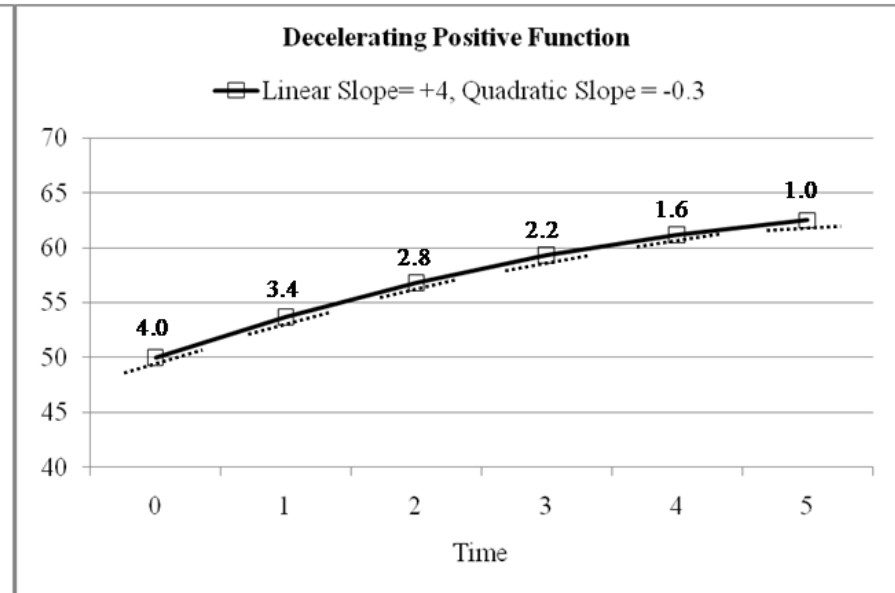
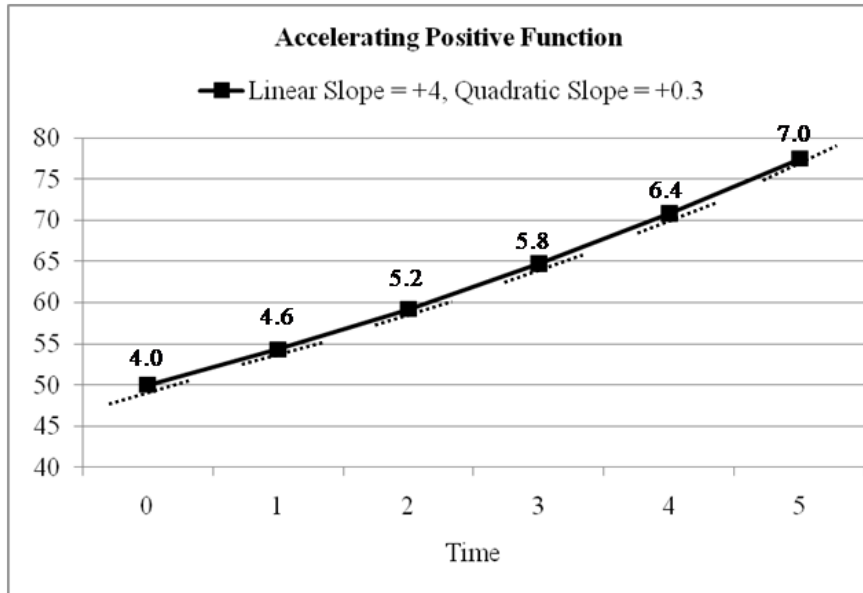
$$\text{Slope of } z_i: \beta_2 + \beta_3(x_i)$$

$$\hat{y}_t = \beta_0 + \beta_1(\text{time}_t) + \beta_3(\text{time}_t)^2$$

$$\text{Slope of } \text{time}_t: \beta_1 + 2\beta_3(\text{time}_t)$$

- Because time is interacting with itself, there is no second “main effect” in the model for the interaction to modify. So the quadratic time slope gets applied twice when added to the one (main) linear time slope

Examples of Fixed Quadratic Time Trends



Conditionality of Polynomial **Fixed Time Effects**

- We've seen how marginal main effect slopes become conditional simple slopes once they are part of an interaction
- The same is true for polynomial **fixed time effects**:
 - **Fixed Intercept Only?**
 - Fixed Intercept = predicted mean of y_{ti} for any occasion (= grand mean)
 - **Add Fixed Linear Time Slope?**
 - Fixed Intercept = **now** predicted mean of y_{ti} at time=0 from linear time trend (would be different if time were centered at a different 0 point)
 - Fixed Linear Time Slope = mean linear rate of change across all occasions (would be the same even if time were centered at a different 0 point)
 - **Add Fixed Quadratic Time Slope?**
 - Fixed Intercept = still predicted mean of y_{ti} at time=0 (but from quadratic time trend; would be different if time were centered at a different 0 point)
 - Fixed Linear Time = **now** mean linear rate of change at time=0 (would now be different if time were centered at a different 0 point)
 - Fixed Quadratic Time = half the mean rate of acceleration or deceleration of linear change over time across all occasions (the linear time slope changes at a constant rate over time)

Polynomial **Fixed** vs. **Random** Time Effects

- **Polynomial fixed effects** combine to describe the mean trajectory over time (can have fixed time slopes up to $n - 1$):
 - Fixed Intercept = Predicted mean of y_{ti} *at time=0*
 - Fixed Linear Time Slope = Mean linear rate of change *at time=0*
 - Fixed Quadratic Time Slope = Half of mean acceleration/deceleration in linear rate of change (2*quad is how the linear time slope changes per unit time if quadratic slope is highest-order fixed effect for time)
- **Polynomial random effects** (individual deviations from the fixed effects) describe individual differences in those change parameters (can have random time slopes up to $n - 2$):
 - Random Intercept = BP variance in predicted mean of y_{ti} *at time=0*
 - Random Linear Time Slope = BP variance in linear time slope *at time=0*
 - Random Quadratic Time Slope = BP variance in **half** the rate of acceleration or deceleration of linear rate of change (over all occasions if quadratic time slope is highest-order random effect for time)

Random Quadratic Time Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}\text{Time}_{ti} + \beta_{2i}\text{Time}_{ti}^2 + e_{ti}$

Level 2 Equations (one per β):

$$\beta_{0i} = \gamma_{00} + u_{0i}$$

β_{0i} ↑ Intercept for person i
 γ_{00} ↑ Fixed (mean) Intercept
 u_{0i} ↑ Random (Deviation) Intercept

$$\beta_{1i} = \gamma_{10} + u_{1i}$$

β_{1i} ↑ Linear Time Slope for person i
 γ_{10} ↑ Fixed (mean) Linear Slope
 u_{1i} ↑ Random (Deviation) Linear Slope

$$\beta_{2i} = \gamma_{20} + u_{2i}$$

β_{2i} ↑ Quadratic Time Slope for person i
 γ_{20} ↑ Fixed (mean) Quad Slope
 u_{2i} ↑ Random (Deviation) Quad Slope

Fixed Effect Subscripts:

1st = which level-1 term
 2nd = which level-2 term

of Possible Time-Related Slopes by # of Occasions (n):

Fixed time slopes = $n - 1$
 # Random time slopes = $n - 2$

Need $n = 4$ occasions to fit random quadratic time model

Example Sequence for Testing Fixed and Random Polynomial Time Slopes

Build up fixed and random time slopes simultaneously:

1. Empty Means, Random Intercept → to calculate ICC
2. Fixed Linear Slope, Random Intercept → check fixed linear p -value
3. Random Linear Slope → check $-2\Delta LL(df \approx 2)$ for random linear slope variance (+1 covariance with random intercept)
4. Fixed Quadratic Slope, Random Linear Time → check fixed quadratic p -value
5. Random Quadratic Slope → check $-2\Delta LL(df \approx 3)$ for quadratic slope variance (+2 covariances with other random effects)
6. (other fixed and random time-related slopes as needed)

*** In general: Can use **REML** for all models, so long as you:

- Test significance of new **fixed** slopes by their Wald test **p -values**
- Test significance of new **random** slopes in separate step by **$-2\Delta LL$**
- Also see if AIC and BIC are smaller when adding random effects

Conditionality of Polynomial Random Effects

- We saw previously that lower-order fixed time slopes are conditional on higher-order polynomial fixed time slopes
- The same is true for polynomial **random time slopes**:
 - **Random Intercept Only?**
 - Random Intercept = BP variance *for any occasion* in predicted mean of y_{ti}
(= variance in grand mean because individual lines are parallel)
 - **Add Random Linear Time Slope?**
 - Random Intercept = **now** BP variance *at time=0* in predicted mean of y_{ti}
(*would be different if time were centered at a different 0 point*)
 - Random Linear Time Slope = BP variance *across all occasions* in linear rate of change
(*would be the same if time were centered at a different 0 point*)
 - **Add Random Quadratic Time Slope?**
 - Random Intercept = still BP variance *at time=0* in predicted mean of y_{ti}
 - Random Linear Time Slope = **now** BP variance *at time=0* in linear rate of change
(*would be different if time were centered at a different 0 point*)
 - Random Quadratic Time Slope = BP variance *across all occasions* in **half** of accel/decel of change
(*would be the same if time were centered at a different 0*)

Random Effects Allowed by #Occasions

	<u>Data</u>		<u>G Matrix</u>	<u>R Matrix</u>	<u>Variance Model # Parameters</u>
<p><u>$n=2$ occasions</u> 3 unique pieces of information</p>	$\begin{bmatrix} \sigma_1^2 & & \\ \sigma_{21} & \sigma_2^2 & \\ & & \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 & \\ & \end{bmatrix}$ <p>Random Intercept only</p>	$\begin{bmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}$	2

<p><u>$n=3$ occasions</u> 6 unique pieces of information</p>	$\begin{bmatrix} \sigma_1^2 & & & \\ \sigma_{21} & \sigma_2^2 & & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \\ & & & \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 & & & \\ & \tau_{U_{01}}^2 & \tau_{U_1}^2 & \\ & & & \end{bmatrix}$ <p>Up to 1 Random slope</p>	$\begin{bmatrix} \sigma_e^2 & 0 & 0 \\ 0 & \sigma_e^2 & 0 \\ 0 & 0 & \sigma_e^2 \end{bmatrix}$	4
---	---	---	--	--	----------

<p><u>$n=4$ occasions</u> 10 unique pieces of information</p>	$\begin{bmatrix} \sigma_1^2 & & & & \\ \sigma_{21} & \sigma_2^2 & & & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & & \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 & \\ & & & & \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 & & & & \\ & \tau_{U_{01}}^2 & \tau_{U_1}^2 & & \\ & & \tau_{U_{02}}^2 & \tau_{U_{12}}^2 & \tau_{U_2}^2 \\ & & & & \end{bmatrix}$ <p>Up to 2 Random slopes</p>	$\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$	7
--	---	---	---	--	----------

Predicted \mathbf{V} Matrix from Polynomial Random Effects Models

- **Random linear time model?** Variance has a **quadratic** dependence on time

- Variance will be at a minimum when time = $-\text{Cov}(U_0, U_1)/\text{Var}(U_1)$, and will increase parabolically and symmetrically over time
- **Predicted variance** at each occasion and covariance between A and B:

$$\text{Var}(y_{\text{time}}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0, U_1)(\mathbf{time}_t) + \text{Var}(U_1)(\mathbf{time}_t^2)$$

$$\text{Cov}(y_A, y_B) = \text{Var}(U_0) + \text{Cov}(U_0, U_1)(A + B) + \text{Var}(U_1)(AB)$$

- **Random quadratic time model?** Variance has a **quartic** dependence on time

$$\text{Var}(y_{\text{time}}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0, U_1)(\mathbf{time}_t) + \text{Var}(U_1)(\mathbf{time}_t^2) + 2\text{Cov}(U_0, U_2)(\mathbf{time}_t^2) + 2\text{Cov}(U_1, U_2)(\mathbf{time}_t^3) + \text{Var}(U_2)(\mathbf{time}_t^4)$$

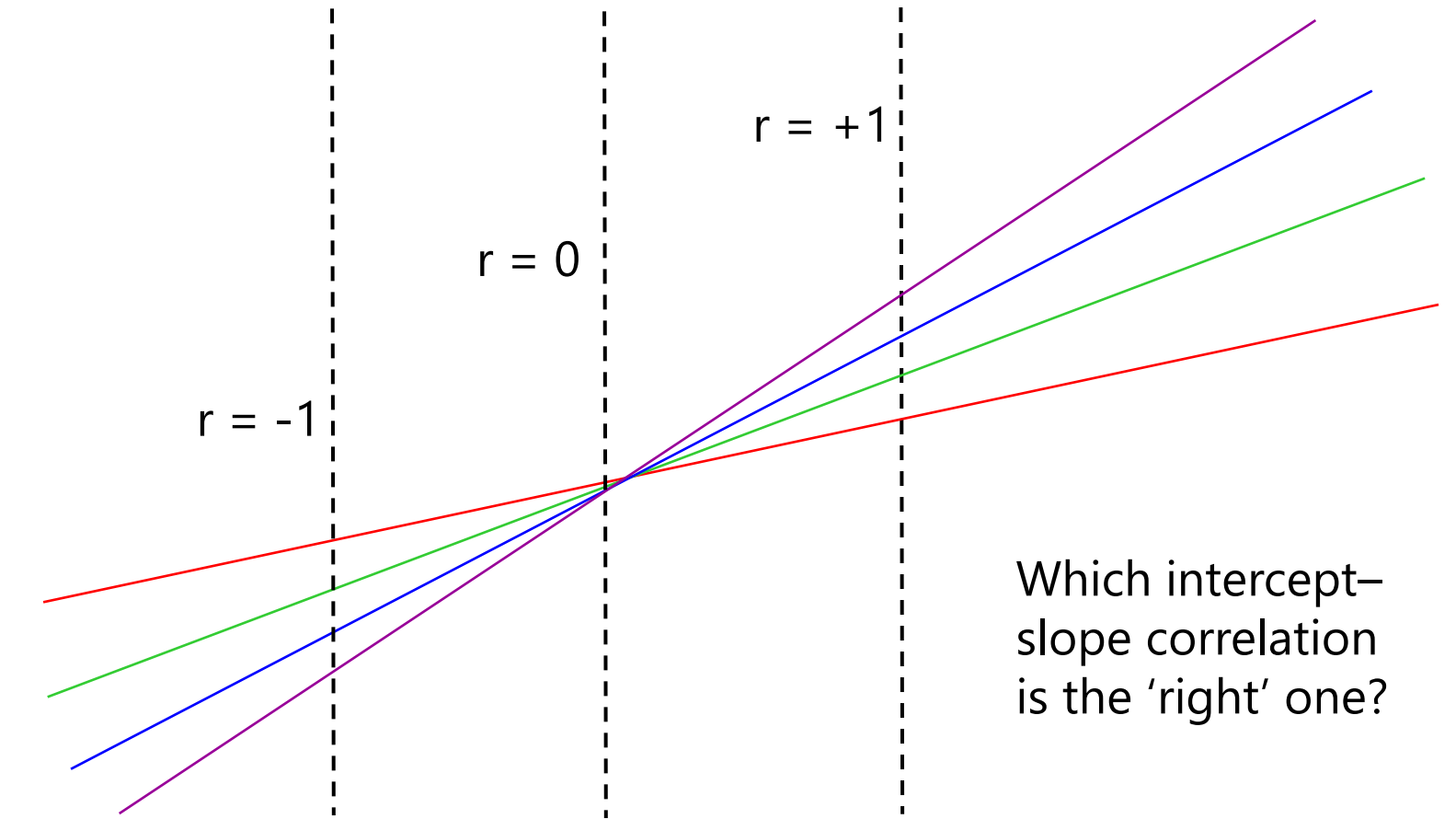
$$\text{Cov}(y_A, y_B) = \text{Var}(U_0) + \text{Cov}(U_0, U_1)(A + B) + \text{Var}(U_1)(AB) + \text{Cov}(U_0, U_2)(A^2 + B^2) + \text{Cov}(U_1, U_2)[(AB^2) + (A^2B)] + \text{Var}(U_2)(A^2B^2)$$

- *The point of the story: random effects of time are a way of allowing the variances and covariances to differ over time in specific, time-dependent patterns (that result from differential individual change over time).*

Rules for Polynomial Time Models (and in general for fixed and random effects)

- On the same side of the model (means or variances side), lower-order effects stay in EVEN IF NONSIGNIFICANT (for correct interpretation)
 - e.g., Significant *fixed* quadratic slope? Keep the *fixed* linear slope
 - e.g., Significant *random* quadratic slope? Keep the *random* linear slope
- You can have a significant random effect EVEN IF the corresponding fixed effect is not significant (keep fixed anyway → what random deviates from):
 - e.g., Fixed linear slope not significant, but random linear slope is significant?
→ No linear change *on average*, but have significant individual differences in change
- Language: A random effect supersedes a fixed effect:
 - If Fixed = intercept, linear, quad; Random = intercept, linear, quad?
 - Call it a "Random quadratic model" (implies all lower-order terms)
 - If Fixed = intercept, linear, quad; Random = intercept, linear?
 - Call it a "Fixed quadratic, random linear model" (distinguishes no random quad)
- Intercept–slope correlation depends largely on centering of time...

Correlation Between Random Intercept and Random Linear Slope Depends on Time=0



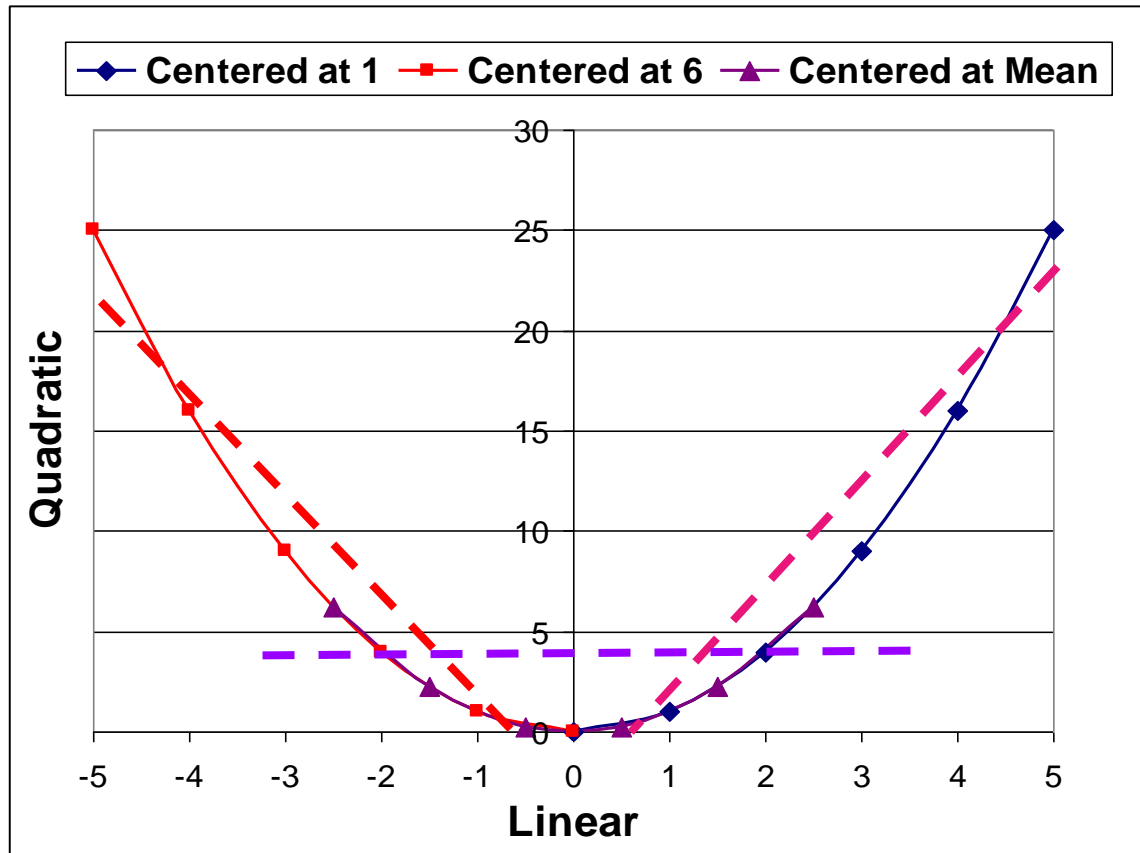
!! Nonparallel lines will eventually cross !!

Correlations Among Polynomial Slopes

Session Centered at 1:		
Session	Linear	Quadratic
1	0	0
2	1	1
3	2	4
4	3	9
5	4	16
6	5	25

Session Centered at 6:		
Session	Linear	Quadratic
1	-5	25
2	-4	16
3	-3	9
4	-2	4
5	-1	1
6	0	0

Session Centered at Mean:		
Session	Linear	Quadratic
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	2.25
6	2.5	6.25



Correlations among polynomial time slopes can be induced by centering time near the start or near the end.

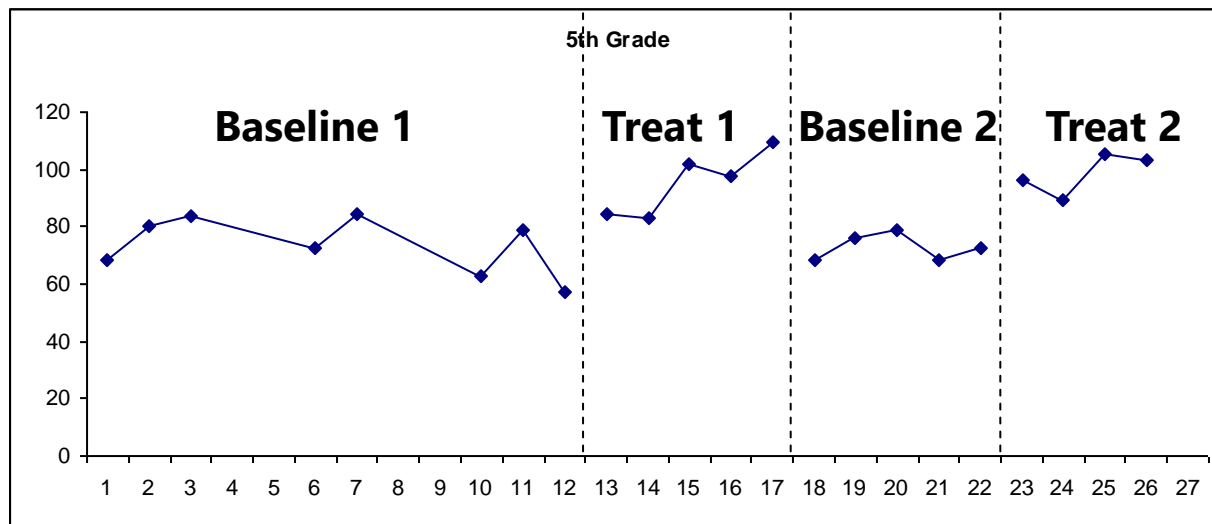
Therefore, these correlations will be **most** interpretable when centering time at its mean instead.

Intermediate Summary

- Modeling within-person change involves specifying slopes related to time for both sides of the model
 - **Fixed slopes in the model for the means:**
 - What kind of change am I observing on average?
 - What kind of trajectory will reproduce those means?
 - **Random slopes in the model for the variance:**
 - What kind of individual differences in change am I observing?
 - How many random slopes do I need to reproduce the observed pattern of variances and covariances over time?
- One option: Polynomial models (linear, quadratic, cubic)
 - Terms work together to describe non-linear trajectories
 - Careful with the covariances among random effects, though
- Coming next: Piecewise slopes and truly nonlinear change...

Other Random Effects Models of Change

- **Piecewise models:** Discrete slopes for discrete phases of time
 - Separate terms describe sections of overall trajectories
 - Useful for examining change in intercepts and slopes before vs. after discrete events (changes in policy, interventions)
 - **Must know where the break point is ahead of time!**
 - Otherwise, you need a “latent change point” model to find the breaks



Piecewise Model Real Example:

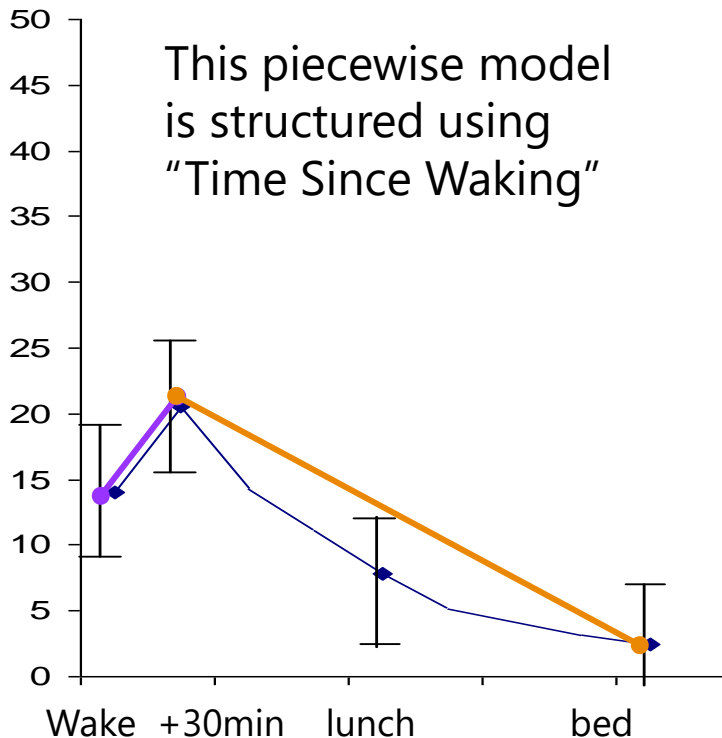
4 slopes
(one per phase)

3 “jumps”
(shift in intercept
between phases)

Example of Daily Cortisol Fluctuation: Morning Rise and Afternoon Decline

Average Trajectories

This piecewise model is structured using "Time Since Waking"



Example SAS code to create two piecewise slopes from continuous time of day in long data:

```
IF occasion=1 THEN DO;
```

```
    P1=0;           P2=0; END;
```

```
IF occasion=2 THEN DO;
```

```
    P1 = time2-time1; P2=0; END;
```

```
IF occasion=3 THEN DO;
```

```
    P1 = time2-time1; P2=time3-time2; END;
```

```
IF occasion=4 THEN DO;
```

```
    P1 = time2-time1; P2=time4-time2; END;
```

Note that a quadratic slope may be necessary for the afternoon decline trajectory!

Random Two-Slope Piecewise Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}\text{Slope1}_{ti} + \beta_{2i}\text{Slope2}_{ti} + e_{ti}$

Level 2 Equations (one per β):

$$\beta_{0i} = Y_{00} + U_{0i}$$

Intercept for person i Fixed (mean) Intercept Random (Deviation) Intercept

$$\beta_{1i} = Y_{10} + U_{1i}$$

Slope1 for person i Fixed (mean) Slope1 Random (Deviation) Slope1

$$\beta_{2i} = Y_{20} + U_{2i}$$

Slope2 for person i Fixed (mean) Slope2 Random (Deviation) Slope2

Fixed Effect Subscripts:

1st = which level-1 term
2nd = which level-2 term

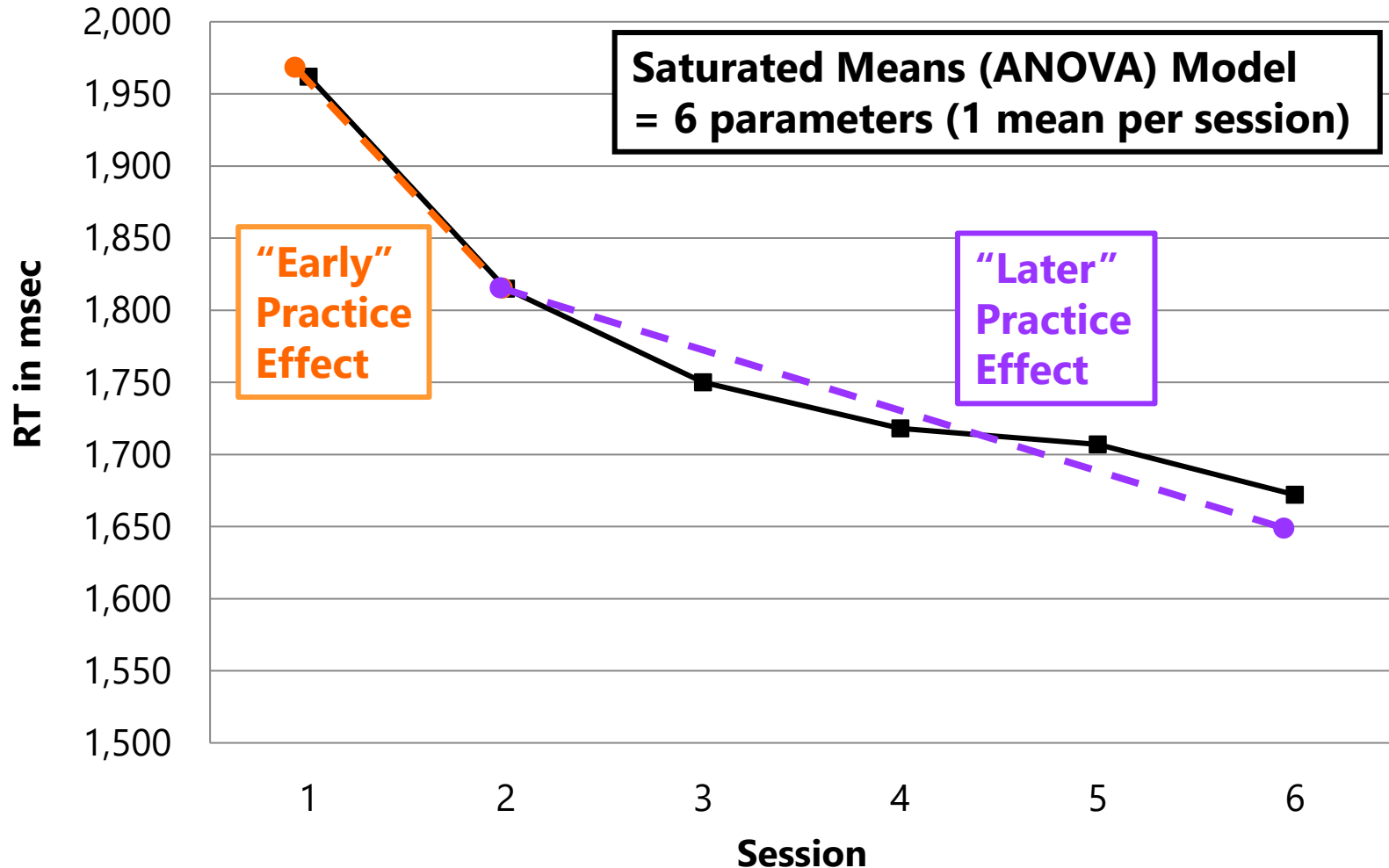
of Possible Time-Related Slopes by # of Occasions (n):

Fixed time slopes = $n - 1$
Random time slopes = $n - 2$

Need $n = 4$ occasions to fit random two-slope model

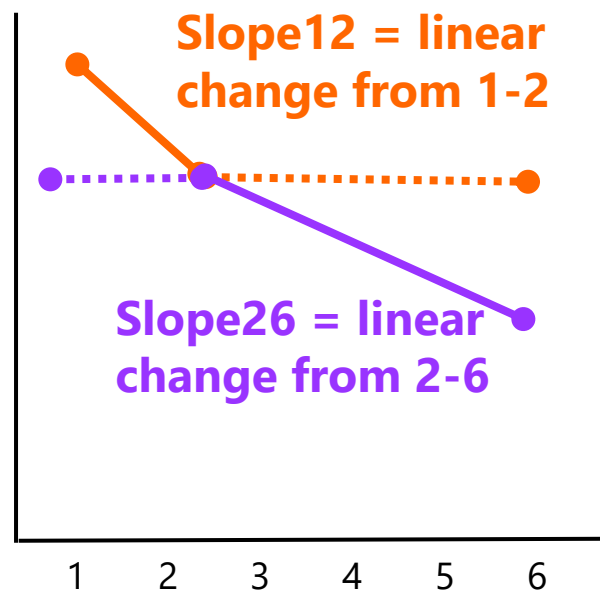
What kind of piecewise model could predict our example data mean change across sessions?

Number Match 3 Mean Response Times by Session



Piecewise Models: Two Direct Slopes

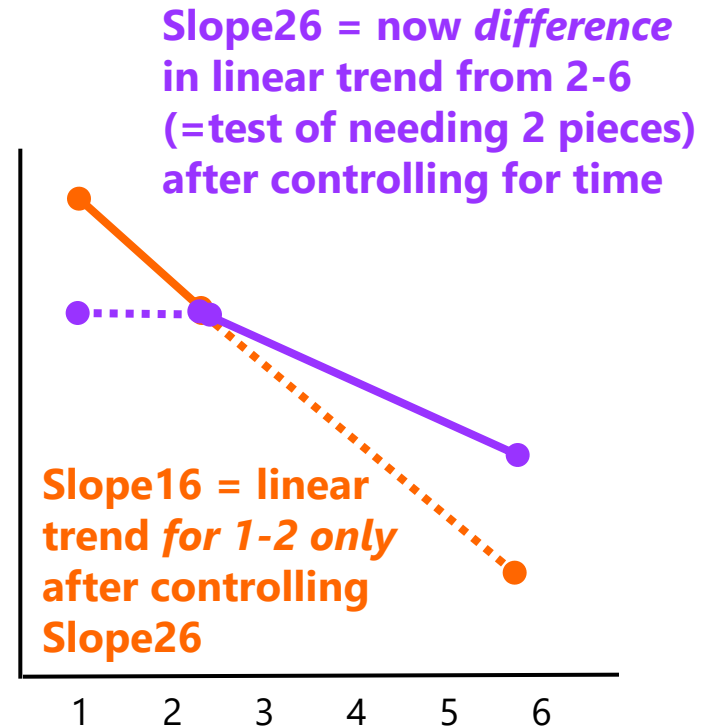
- “Early Practice Slope” and “Later Practice Slope”
- Use to specify slopes through each discrete phase directly
 - Dotted lines → without other slope
- Session (1 to 6) gets recoded into two new time predictor variables, as shown below:



Session	1	2	3	4	5	6
Early Practice → Slope12 =	0	1	1	1	1	1
Later Practice → Slope26 =	0	0	1	2	3	4

Piecewise Models: Slope + Deviation Slope

- “Linear Time Slope” and “Deviation Slope”
- Use to test if multiple slopes are needed in the first place!
 - Dotted lines → without other slope
- Initial slope predictor is coded differently, second slope predictor is same:



Session		1	2	3	4	5	6
Time	→ Slope16 =	0	1	2	3	4	5
Deviation	→ Slope26 =	0	0	1	2	3	4

2 Direct Slopes Model: Random Effects

- Parameters directly **represent each part** of trajectory:
- **Fixed effects** (3) for mean change over sessions
 - Fixed Intercept = expected mean y_{ti} when both slopes = 0
(at session 1 given slopes coding)
 - Fixed Slope12 = expected linear rate of change from 1 to 2
 - Fixed Slope26 = expected linear rate of change from 2 to 6
- Leads to possible **random effects** (3 var + 3 covar)
 - Random Intercept = BP variance in expected mean y_{ti}
when both slopes = 0 (at session 1)
 - Random Slope12 = BP variance in linear slope from 1 to 2
 - Random Slope26 = BP variance in linear slope from 2 to 6

Slope + Deviation Slope: Random Effects

- Parameters directly **differences across parts** of trajectory:
- **Fixed effects** (3) for mean change over sessions
 - Fixed Intercept = expected mean y_{ti} when both slopes = 0
(at session 1 given slopes coding)
 - Fixed Slope16 = expected linear rate of change from 1 to 2
(after controlling for slope26)
 - Fixed Slope26 = expected **extra** linear rate of change from 2 to 6
(after controlling for slope16, which is just time)
- Leads to possible **random effects** (up to 3 var+3 cov)
 - Random Intercept = BP variance in expected mean y_{ti}
when both slopes = 0 (at session 1)
 - Random Slope16 = BP variance in linear slope from 1 to 2
 - Random Slope26 = BP variance in **extra** linear slope from 2 to 6

Saturated Means via Piecewise Slopes Models

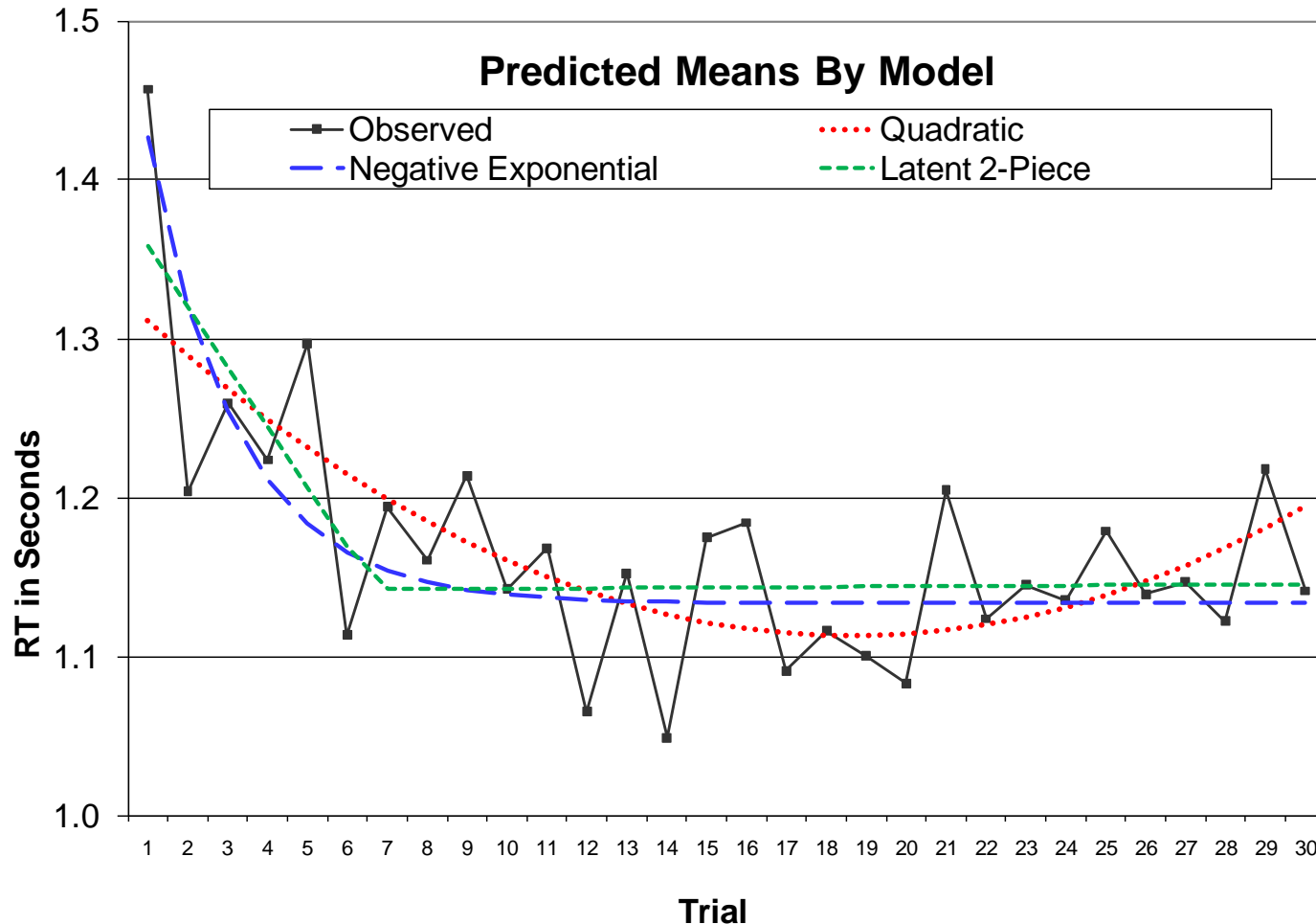
- You can fit **fixed** piecewise time slopes up to $n-1$, but only **random** piecewise time slopes up to $n-2$
 - 3 occasions? Up to 2 fixed pieces, but only 1 random piece
 - 4 occasions? Up to 3 fixed pieces, but only 2 random pieces
 - $n-1$ fixed slopes will perfectly reproduce observed means
- Given this constraint (and balanced data), you may need an ACS model instead of random slopes in variance model:
 - Example: $n=3 \rightarrow$ model for the means = 2 fixed piecewise slopes for time; model for the variance could be...
 - UN, CSH, CS (Random Intercept Only), Random Intercept + Random Slope₁₂, OR Random Intercept + Random Slope₂₃
 - Everything is nested within UN; can also use AIC and BIC to choose

Summary: Piecewise Slopes Models

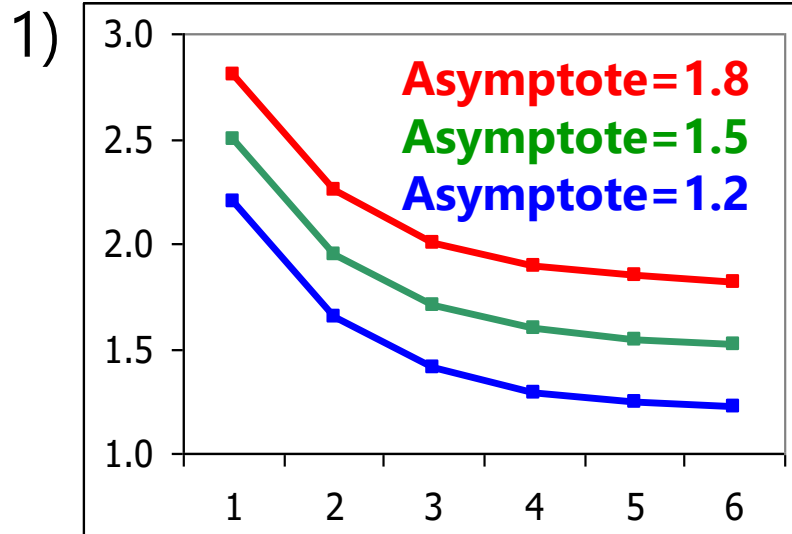
- **Piecewise models are useful for discontinuous trajectories** (empirically or based on the study design)
 - Use slope + deviation slope(s) version to test if > 1 slope is necessary
- If all effects are random, the slope + deviation slope and the direct slopes versions of the models will be equivalent
 - Select the one that has the random effects variance you want to predict
- Keep all the pieces in the model (even if non-significant) in order to maintain a correct interpretation of each
- Each piecewise slope can be linear or non-linear as needed
 - e.g., $\text{piece1} + \text{piece2} + \text{piece2}^2 \rightarrow$ linear, then non-linear trajectory
- You may also want to test for a 'drop' or 'jump' in intercept at the break point in addition to change in slope, data permitting
- You may also want to introduce nonlinear slopes into each piece (see [Harring et al. 2021](#) for some examples)

Other Random Effects for Change

- **“Truly” nonlinear models:** Non-additive terms to describe change
 - Models can include **asymptotes** (so change can “shut off” as needed)
 - Include **power** and **exponential** functions (see chapter 6 for references)



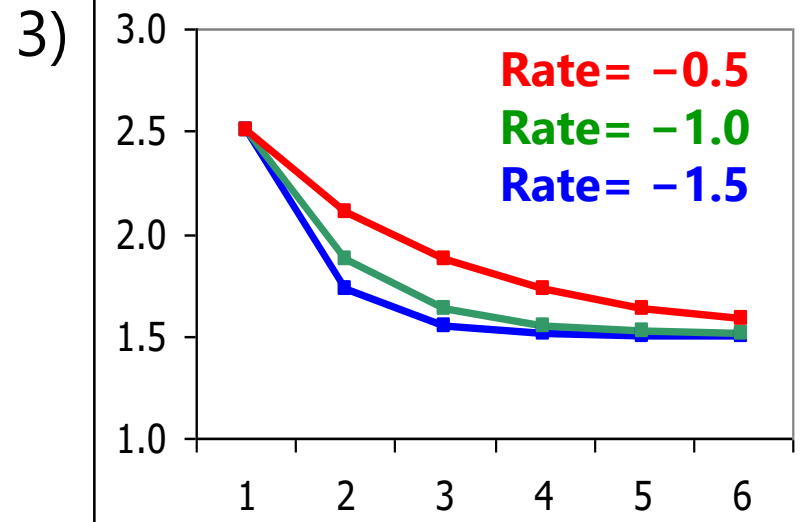
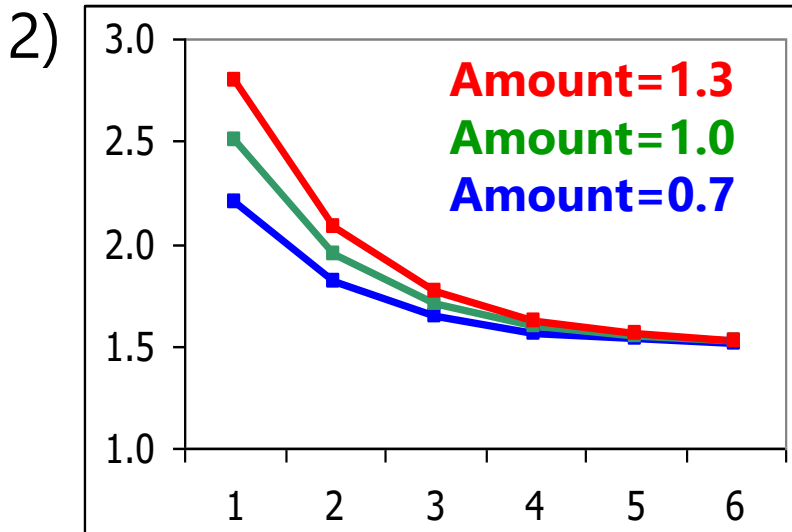
(Negative) Exponential Model Parameters



1) Different **Asymptotes**, same amount and rate

2) Different **Amounts**, same asymptote and rate

3) Different **Rates**, same asymptote and amount



Exponential Models of Change over Time

- The name **positive or negative** reflects whether the data are moving **away or towards** asymptote
 - *Accelerating* trajectory (up or down) = “positive” exponential
 - *Decelerating* trajectory (up or down) = “negative” exponential
- Amount reflects distance from asymptote to time 0, multiplied by $\exp(\text{rate} \times \text{time})$
 - Decrease across time to asymptote = positive amount
 - Increase across time to asymptote = negative amount
- Amount can also be replaced by an intercept
 - Asymptote + Amount = Intercept
- Cannot be estimated in standard MLM software given its nonlinear parameters (I use SAS PROC NLMIXED instead)

Exponential Model (with 3 Random Effects)

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i} * \exp(\beta_{2i} * \text{Time}_{ti}) + e_{ti}$

Level 2 Equations (one per β):

$$\beta_{0i} = Y_{00} + U_{0i}$$

Asymptote for person i Fixed (mean) Asymptote Random (Deviation) Asymptote

$$\beta_{1i} = Y_{10} + U_{1i}$$

Amount for person i Fixed (mean) Amount Random (Deviation) Amount

$$\beta_{2i} = Y_{20} + U_{2i}$$

Rate for person i Fixed (mean) Rate Random (Deviation) Rate

Fixed Effect Subscripts:

1st = which level-1 term
2nd = which level-2 term

of Possible Time-Related Slopes by # of Occasions (n):

Fixed time slopes = $n - 1$
Random time slopes = $n - 2$

Also need 4 occasions to fit random exponential model

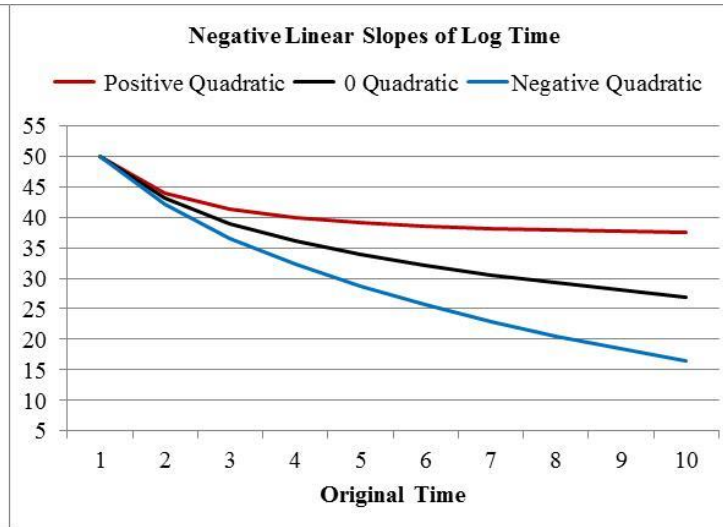
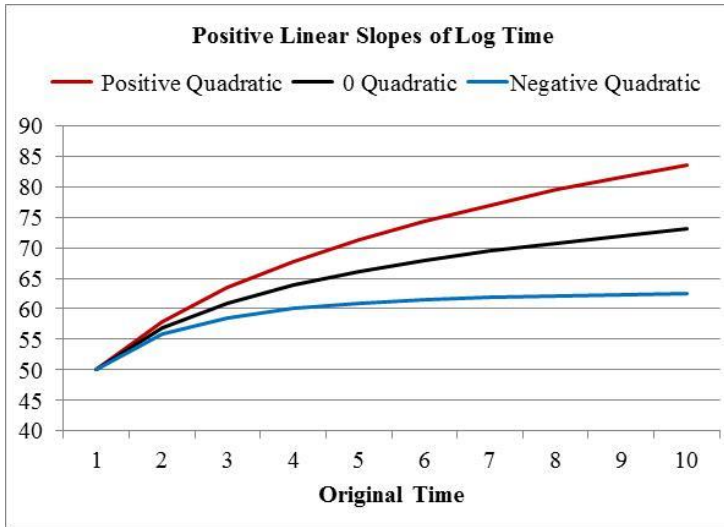
(Likely need way more occasions to find U_{2i} , though)

Estimating Truly Nonlinear Models

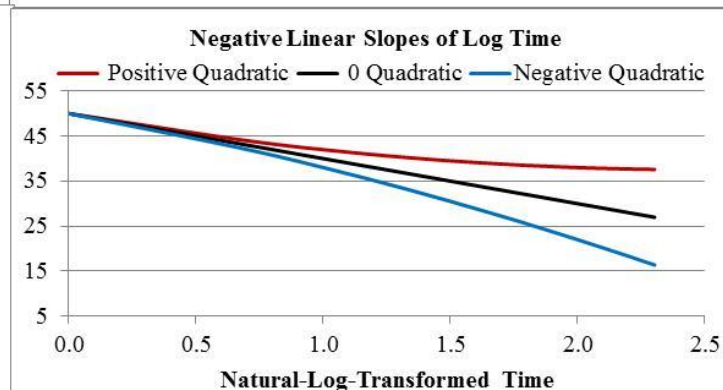
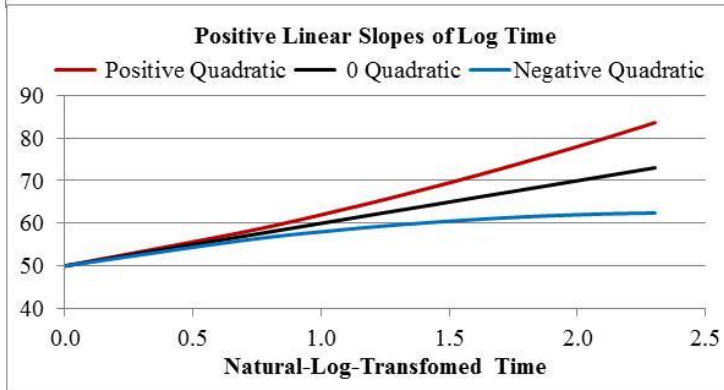
- Not all forms of change fit polynomial models
 - Quadratic models imply that what goes up must come down (and vice-versa), which doesn't often fit patterns of change
 - Change frequently needs to "shut off" (at 1 or 2 asymptotes)
- Many kinds of truly nonlinear models can be used for predicting trajectories of change over time
 - Linear in variables vs. linear in parameters (exp \rightarrow nonlinear)
 - Logistic, power, exponential... see end of chapter 6 for ideas
- Require extra steps to evaluate estimation quality
 - Start values are needed, especially for random variances
 - Check that "gradient" values are as close to 0 as possible (partial first derivative of that parameter in LL function)
 - Only ML is available, so what to do in small samples???

How to Mimic an Exponential Model

If you need to use REML, a predictor of natural-log-transformed time may be a decent substitute for a truly nonlinear model



A linear slope of log time (black lines) predicts an exponential curve across *original* time.



Quadratic slopes of log time (red or blue lines) can speed up or slow down the curve.

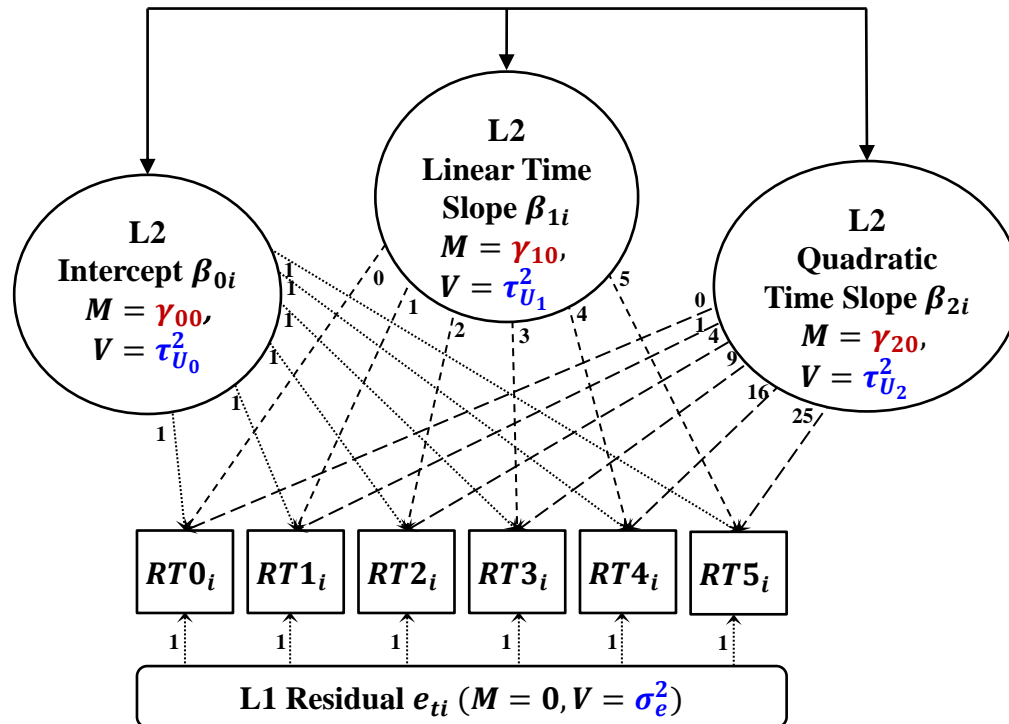
Bottom: There is a linear relationship between log-time and the outcome.

Which change family should I choose?

- Within a given family, nested models can usually be compared to judge the need for each parameter
 - e.g., linear vs. quadratic? One slope vs. two slopes?
 - Usual nested model comparison rules apply (Wald test p -values for fixed effects, $-2\Delta LL$ test p -values for random effects)
 - When using REML and you have balanced data, you can test the absolute fit of each side separately to see if you are “there yet”
- Between families, however, alternative models of change may not be nested, so deciding among them can be tricky
 - e.g., quadratic vs. two-slope vs. log time vs. exponential?
 - Use ML AIC and BIC to see what is “preferred” across the families
 - In balanced data only, you can also compare each alternative to a saturated means, UN model (the “H1”) using ML as test of absolute fit
 - If doing so using SEM, you need [a different “null” model](#) for computing CFI/TLI
 - Also consider plausibility of alternative models in terms of both data predictions and theoretical predictions (e.g., does change last forever?)

Bonus: Change Models using SEM

- All the linear model options (polynomials, piecewise, log-time) could also be estimated using single-level SEM using a wide data format
 - Pros: Random effects can be predictors; change in latent variables instead
 - Cons: Bad for small samples (no REML estimation and no denominator DF)
- For example, here is a quadratic change model using SEM:



Level 1:

$$RT_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}$$

Level 2 Equations (one per β):

$$\beta_{0i} = \gamma_{00} + U_{0i}$$

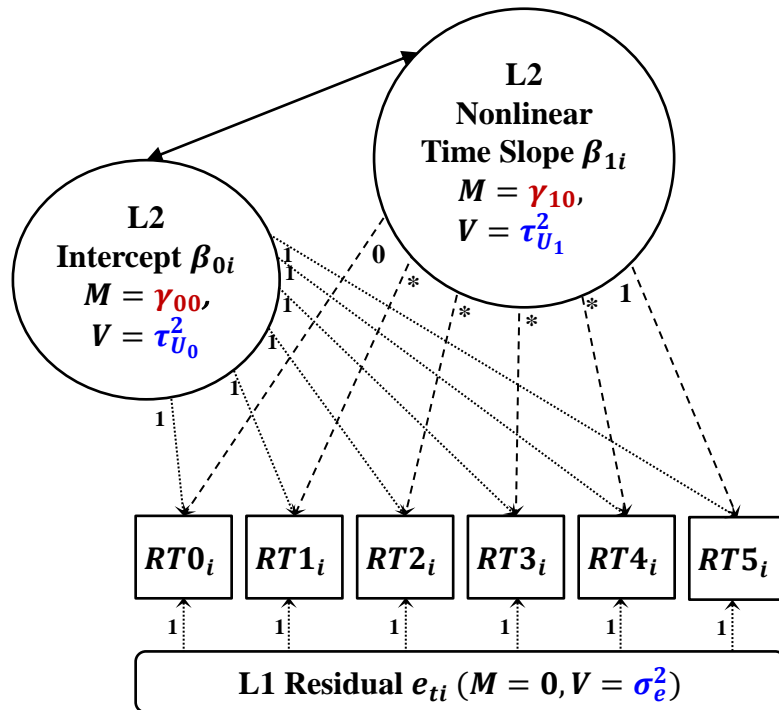
$$\beta_{1i} = \gamma_{10} + U_{1i}$$

$$\beta_{2i} = \gamma_{20} + U_{2i}$$

Other shapes of change can be created by modifying the loadings (use Mplus TSCORES for unbalanced time)...

... but what if you let the data decide?

Estimated Nonlinearity via Latent Basis



Btw, fixed loadings imply the same shape holds for everyone, but the loadings *could* be random instead—see [McNeish \(2020\)](#)

- Latent basis models only require one slope to capture nonlinearity
 - Can be helpful when slopes become predictors (no “linear vs quadratic”)
- How? Fix two factor loadings, estimate the rest—for example:
 - 0...1 → loadings indicate proportion of total change at each occasion
 - 0 ... Last occasion → loadings indicate amount of change at each occasion in the same metric as other models (intercept at beginning)
 - First occasion ... 0 → loadings indicate amount of change at each occasion in the same metric as other models (intercept at end instead)

Example: Mplus Single-Level SEM Syntax

Just showing **MODEL** part, which would be preceded by **DATA**, **VARIABLE**, and **ANALYSIS** as usual (estimated using **wide** data)

```
!!!! Random quadratic model of change

! Factor loadings fixed by @
  Int BY RT0@1 RT1@1 RT2@1 RT3@1 RT4@1 RT5@1;
  Lin BY RT0@0 RT1@1 RT2@2 RT3@3 RT4@4 RT5@5;
  Qua BY RT0@0 RT1@1 RT2@4 RT3@9 RT4@16 RT5@25;

! Factor intercepts estimated = fixed effects
  [Int Lin Qua];
! Level-2 factor variances estimated (in G)
  Int Lin Qua;
! Level-2 factor covariances estimated (in G)
  Int Lin Qua WITH Int Lin Qua;

! Per-occasion intercepts fixed to 0
  [RT0@0 RT1@0 RT2@0 RT3@0 RT4@0 RT5@0];

! Level-1 residual variances held equal (in R)
  RT0 RT1 RT2 RT3 RT4 RT5 (ResVar);
```

```
!!!! Random latent basis model of change

! Factor loadings fixed by @
  Int BY RT0@1 RT1@1 RT2@1 RT3@1 RT4@1 RT5@1;
  Slp BY RT0@0 RT1* RT2* RT3* RT4* RT5@1;
! Loadings estimated as 0.57, 0.76, 0.90, 0.98

! Factor intercepts estimated = fixed effects
  [Int Slp];
! Level-2 factor variances estimated (in G)
  Int Slp;
! Level-2 factor covariance estimated (in G)
  Int WITH Slp;

! Per-occasion intercepts fixed to 0
  [RT0@0 RT1@0 RT2@0 RT3@0 RT4@0 RT5@0];

! Level-1 residual variances held equal (in R)
  RT0 RT1 RT2 RT3 RT4 RT5 (ResVar);
```

Note: There are Mplus syntax shortcuts for growth models I am not using: (1) to be explicit about what the model contains, (2) to not estimate separate residual variances

Example: R Single-Level SEM Syntax

```
RandQuadSyntax = "  
# Factor loadings fixed by *  
Int =~ 1*RT0 + 1*RT1 + 1*RT2 + 1*RT3 + 1*RT4 + 1*RT5  
Lin =~ 0*RT0 + 1*RT1 + 2*RT2 + 3*RT3 + 4*RT4 + 5*RT5  
Qua =~ 0*RT0 + 1*RT1 + 4*RT2 + 9*RT3 + 16*RT4 + 25*RT5  
  
# Factor intercepts estimated = fixed effects  
Int ~ 1; Lin ~ 1; Qua ~ 1  
# Level-2 factor variances estimated (in G)  
Int ~~ Int; Lin ~~ Lin; Qua ~~ Qua  
# Level-2 factor covariances estimated (in G)  
Int ~~ Lin + Qua; Lin ~~ Qua  
  
# Per-occasion intercepts fixed to 0  
RT0 ~ 0; RT1 ~ 0; RT2 ~ 0  
RT3 ~ 0; RT4 ~ 0; RT5 ~ 0  
  
! Level-1 residual variances held equal (in R)  
RT0 ~~ (ResVar)*RT0; RT1 ~~ (ResVar)*RT1  
RT2 ~~ (ResVar)*RT2; RT3 ~~ (ResVar)*RT3  
RT4 ~~ (ResVar)*RT4; RT5 ~~ (ResVar)*RT5  
"  
RQModel = lavaan(data=Example2wide,  
                 model=RandQuadSyntax,  
                 estimator="ML", mimic="mplus")  
summary(RQModel, fit.measures=TRUE, rsquare=TRUE,  
        standardized=TRUE)
```

```
LatentBasisSyntax = "  
# Factor loadings fixed by *  
Int =~ 1*RT0 + 1*RT1 + 1*RT2 + 1*RT3 + 1*RT4 + 1*RT5  
Slp =~ 0*RT0 + RT1 + RT2 + RT3 + RT4 + 1*RT5  
# Loadings estimated as 0.57, 0.76, 0.90, 0.98  
  
# Factor intercepts estimated = fixed effects  
Int ~ 1; Slp ~ 1  
# Level-2 factor variances estimated (in G)  
Int ~~ Int; Slp ~~ Slp  
# Level-2 factor covariances estimated (in G)  
Int ~~ Slp  
  
# Per-occasion intercepts fixed to 0  
RT0 ~ 0; RT1 ~ 0; RT2 ~ 0  
RT3 ~ 0; RT4 ~ 0; RT5 ~ 0  
  
! Level-1 residual variances held equal (in R)  
RT0 ~~ (ResVar)*RT0; RT1 ~~ (ResVar)*RT1  
RT2 ~~ (ResVar)*RT2; RT3 ~~ (ResVar)*RT3  
RT4 ~~ (ResVar)*RT4; RT5 ~~ (ResVar)*RT5  
"  
LBModel = lavaan(data=Example2wide,  
                 model=LatentBasisSyntax,  
                 estimator="ML", mimic="mplus")  
summary(LBModel, fit.measures=TRUE, rsquare=TRUE,  
        standardized=TRUE)
```

Note: There are lavaan syntax shortcuts for growth models I am not using: (1) to be explicit about what the model contains, (2) to not estimate separate residual variances