Review of Fixed Effects within General Linear Models (and especially interaction terms)

- Topics:
  - Fixed slopes: Interpretation and significance
  - Scaling predictor variables: Centering and coding
    - Categorical predictors: Manual vs. program-automated coding
    - Semi-continuous predictor coding: If and how much (piecewise/spline)
    - Testing multiple slopes (for a single predictor or multiple predictors)
  - Linear models with interaction terms
    - Taxonomy terminology: Bivariate marginal, unique marginal, or unique conditional fixed slopes
    - Interpreting interaction slopes as modifiers of main effect slopes

Naming Conventions in the GLM

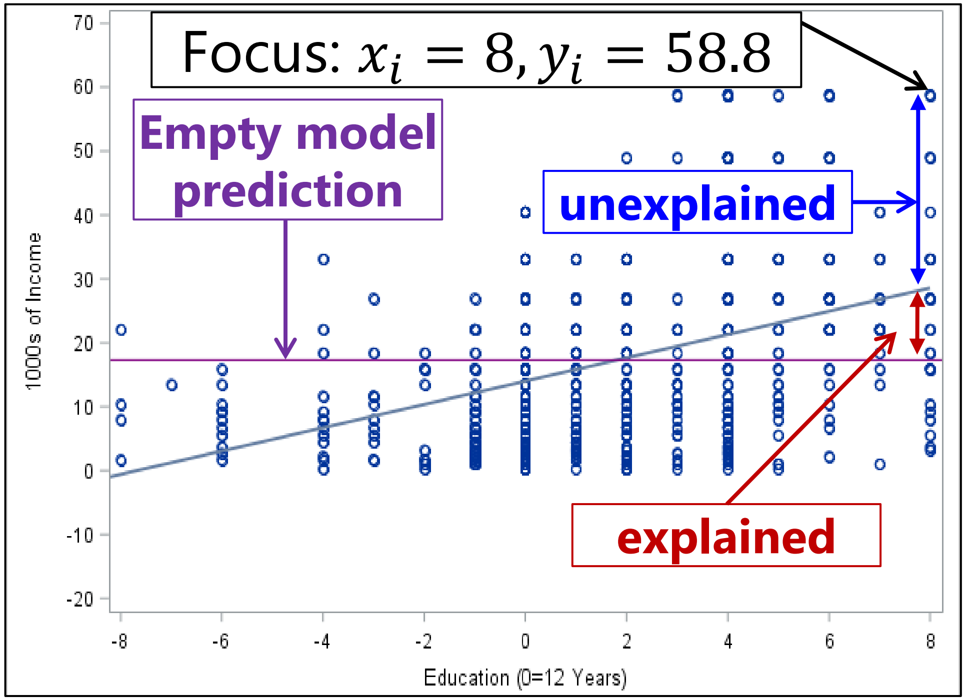
- The **general linear model** incorporates many different labels of single-level analyses (for **independent** obs) under 1 unifying term:

	Categorical Predictors	**Quantitative Predictors**	**Both Types of Predictors**
Univariate (one outcome)	"ANOVA"	"Regression"	"ANCOVA"
Multivariate (2+ outcomes)	"MANOVA"	"Multivariate Regression"	"MANCOVA"

- What these models all have in common is the use of a normal conditional distribution (i.e., for the *residuals* that remain after creating conditional outcomes from the model predictors)
- Btw, predictors do NOT have distributional assumptions!
- The use of these words **almost always implies estimation using** "**least squares**" (LS), *aka* "**ordinary least squares**" (OLS)

A One-Slope GLM Example

The $\boldsymbol{\beta}$ formulas result from the goal of minimizing the squared residuals across the sample—this is called "**ordinary least squares estimation**"—let's see what happens for one example person



<u>**Empty Model**</u> for y_i = income:

$$\boldsymbol{y_i = \beta_0 + e_i}$$

$$\boldsymbol{\hat{y}_{Focus} = 17.3}$$

$$\boldsymbol{y_{Focus} = 17.3 + 41.5}$$

Variance: $\boldsymbol{\sigma_e^2} = \frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N-1} = 190.2$

→ 190.2 is **all** the y_i variance

<u>Add Education as Predictor:</u>

$$\boldsymbol{y_i = \beta_0 + \beta_1(Educ_i - 12) + e_i}$$

$$\boldsymbol{\hat{y}_{Focus} = 14.0 + 1.8(8) = 28.4}$$

$$\boldsymbol{y_{Focus} = 28.4 + 30.4}$$

Variance: $\boldsymbol{\sigma_e^2} = \frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N-2} = 162.3$

→ 162.3 is **leftover** y_i variance

General Linear Models, More Generally

- A **General Linear Model (GLM*)** for outcome $\boldsymbol{y_i}$ looks like this:
  - actual $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{x1_i}) + \boldsymbol{\beta_2}(\boldsymbol{x2_i}) + \cdots \boldsymbol{\beta_p}(\boldsymbol{xp_i}) + \boldsymbol{e_i}$
  - predicted $\hat{\boldsymbol{y}}_{\boldsymbol{i}} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{x1_i}) + \boldsymbol{\beta_2}(\boldsymbol{x2_i}) + \cdots \boldsymbol{\beta_p}(\boldsymbol{xp_i})$

  - The "i" subscript denotes **variables** (that are individual-specific)
  - The $\boldsymbol{\beta}$ ("beta") terms are the model **fixed effects** → **constants** whose subscripts range from 0 up to p as the last fixed effect):
    - $\boldsymbol{\beta_0}$ = **intercept** = expected $\boldsymbol{y_i}$ when all x_i predictors are 0
    - $\boldsymbol{\beta_1}$ = **slope of $\boldsymbol{x1_i}$** = difference in $\boldsymbol{y_i}$ per one-unit difference in $\boldsymbol{x1_i}$
    - $\boldsymbol{\beta_2}$ = **slope of $\boldsymbol{x2_i}$** = difference in $\boldsymbol{y_i}$ per one-unit difference in $\boldsymbol{x2_i}$

    …
    - $\boldsymbol{\beta_p}$ = **slope of $\boldsymbol{xp_i}$** = difference in $\boldsymbol{y_i}$ per one-unit difference in $\boldsymbol{xp_i}$

* GLM may also stand for Generalized Linear Models, which includes General as one type (ugh)

Significance Tests of Fixed Slopes

- Each $\boldsymbol{\beta}$ **fixed slope** has 6 relevant characteristics (*essential to report):
  - ***Estimate** = best guess for the fixed slope from our data (ML→ tallest answer)
  - ***Standard Error** = $\boldsymbol{SE}$ = average distance of sample slope from population slope → expected *inconsistency* of slope across samples
  - $\boldsymbol{t}$**-value** = (Estimate − H_0) / SE = test-statistic for fixed slope against $H_0 (= 0)$
  - **Denominator DF** = $N - k$ (where k = total number of fixed effects)
  - $\boldsymbol{p}$**-value** = (two-tailed) probability of fixed slope estimate *as or more extreme* IF H_0 is true → how unexpected our result is on a t-distribution with 0=H_0, SD=SE
  - **(95%) Confidence Interval** = $\boldsymbol{CI} = Estimate \pm t_{critical} * SE$ = range in which true (population) value of estimate is expected to fall across 95% of samples

- Compare $\boldsymbol{t}$ test-statistic to t critical-value at pre-chosen level of significance (where % unexpected = alpha level): this is a "**univariate Wald test**"
  - Btw, if denominator DF are not used, then $\boldsymbol{t}$ is treated as a $\boldsymbol{z}$ instead (same value)
  - Btw, whether the p-value is found using a $\boldsymbol{t}$**- or** $\boldsymbol{z}$**-distribution** will differ by program and variant in generalized linear models…

Significance of Each Fixed Slope

- Standard Error (SE) for fixed effect estimate β_X in a one-predictor model (remember, SE is like the SD of the estimated parameter):

$$SE_{\beta_X} = \sqrt{\frac{\text{residual variance of Y}}{\text{Var(X)}*(N-k)}}$$

N = sample size
k = number of fixed effects

- When more than one predictor is included, SE turns into:

$$SE_{\beta_X} = \sqrt{\frac{\text{residual variance of Y}}{\text{Var(X)}*(\mathbf{1-R_X^2})*(N-k)}}$$

R_X^2 = X variance accounted for by other predictors, so
$1-R_X^2$ = unique X variance

- So all things being equal, SE is smaller when:
  - More of the outcome variance has been reduced (better predictive model)
    - This means fixed effects can become significant later if R^2 is higher then
  - The predictor has less covariance with other predictors
    - Best case scenario: X is uncorrelated with all other predictors

- If SE is smaller → t-value or z-value is bigger → p-value is smaller

Scaling of Predictor Variables

- Get in the habit of rescaling all **predictors** so **0 is meaningful value**
  - **Why?** To maintain a **meaningful intercept** in ALL models
  - For meaningful **conditional slopes** within interactions (stay tuned)
  - (To avoid estimation problems in multilevel models with random slopes)

- For **quantitative** predictors, this is called (constant) "**centering**"
  - Center by subtracting a constant: sample mean is a common choice, but any meaningful value is good (e.g., known reference, minimum)

- For **categorical** predictors, this is called "**coding**"
  - Create $\boldsymbol{C-1}$ **slopes** to describe $\boldsymbol{C}$ **categories** using values of 0 or 1 ("dummy coding") or values of 0, 1, -1 ("effect coding") in a pattern that creates the desired interpretation of group differences
    - Will perfectly re-create all category means and mean differences using either **fixed effects directly** or **linear combinations** of fixed effects
    - I prefer **dummy coding**, in which 1 chosen category is the "**reference**" for which **all predictors = 0** (instead of reference = overall mean)

Coding Strategies for Categorical Predictors

Indicator coding: Each non-ref category has a 1 value in **1 predictor only** to represent its mean difference from reference (good for **nominal**)

Group	(Inter-cept): A mean	AvsB: Diff for A vs B	AvsC: Diff for A vs C
A	1	0	0
B	1	1	0
C	1	0	1

Either way, all possible category means and mean differences not directly provided by the model fixed effects can be found from linear combinations of them…

Sequential coding: Each non-ref category can have multiple 1 values → predictors then give mean differences between sequential categories (good for **ordinal**)

Happy	(Intercept): 1 Mean	h1v2: 1→2 Diff	h2v3: 2→3 Diff	h3v4: 3→4 Diff	h4v5: 4→5 Diff
1	1	0	0	0	0
2	1	1	0	0	0
3	1	1	1	0	0
4	1	1	1	1	0
5	1	1	1	1	1

Sequential coding can be used to test whether an ordinal predictor can be treated as interval—whether it has a linear slope in predicting an outcome—by testing differences between the sequential slopes

Categorical Predictors: Manual Indicator Coding

- Model: $\boldsymbol{y_i = \beta_0 + \beta_1(d1_i) + \beta_2(d2_i) + \beta_3(d3_i) + e_i}$
  - Variable called "**group**": Control=0, Treat1=1, Treat2=2, Treat3=3
  - New predictors we must create for the model:
    d1 = 0, 1, 0, 0 → difference between Control and Treat1
    d2 = 0, 0, 1, 0 → difference between Control and Treat2
    d3 = 0, 0, 0, 1 → difference between Control and Treat3
  - These interpretations only hold if all three new predictors are included!

- How does the model give us **all possible group differences**?
  By determining each group's mean, and then the difference…

Control Mean (Reference)	Treatment 1 Mean	Treatment 2 Mean	Treatment 3 Mean
$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_0+\beta_1(d1_i)}$	$\boldsymbol{\beta_0+\beta_2(d2_i)}$	$\boldsymbol{\beta_0+\beta_3(d3_i)}$

- Model directly provides 3 mean differences (control vs. each treatment), and indirectly provides another 3 mean differences (differences between treatments) as **linear combinations**… let's see how this works

Categorical Predictors: Manual Indicator Coding

- Model: $y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{d1_i}) + \boldsymbol{\beta_2}(\boldsymbol{d2_i}) + \boldsymbol{\beta_3}(\boldsymbol{d3_i}) + \boldsymbol{e_i}$

Control Mean (Reference)	**Treatment 1 Mean**	**Treatment 2 Mean**	**Treatment 3 Mean**
$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_1}(\boldsymbol{d1_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_2}(\boldsymbol{d2_i})$	$\boldsymbol{\beta_0}+\boldsymbol{\beta_3}(\boldsymbol{d3_i})$

	Alt Group		Ref Group	Difference
Control vs. T1 =	$(\beta_0+\beta_1)$	$-$	(β_0)	$= \beta_1$
Control vs. T2 =	$(\beta_0+\beta_2)$	$-$	(β_0)	$= \beta_2$
Control vs. T3 =	$(\beta_0+\beta_3)$	$-$	(β_0)	$= \beta_3$
T1 vs. T2 =	$(\beta_0+\beta_2)$	$-$	$(\beta_0+\beta_1)$	$= \beta_2 - \beta_1$
T1 vs. T3 =	$(\beta_0+\beta_3)$	$-$	$(\beta_0+\beta_1)$	$= \beta_3 - \beta_1$
T2 vs. T3 =	$(\beta_0+\beta_3)$	$-$	$(\beta_0+\beta_2)$	$= \beta_3 - \beta_2$

2 Ways to Include Categorical Predictors

1. **Manually create** and include dummy-coded predictors
   - Need $C-1$ predictors for C categories, added all at once, **treated as quantitative** (WITH in SPSS, by default in SAS and R, c. in STATA)
   - **We are going to do it this way**, in part because it corresponds directly to a linear model representation → transparency!
   - You have complete control of what your predictors represent!

2. **Let the program** create and include predictors for you
   - **Treated as categorical**: BY in SPSS, CLASS in SAS, i. in STATA, factor in R
     - SPSS and SAS: reference = highest/last; STATA/R: reference = lowest/first
   - Can be more convenient in GLMs to get predicted means if you have many categories, want many differences, or have interactions among categorical predictors—but not in all generalized linear models
   - And it marginalizes over other program-categorical predictors for their main effect F-tests, creating two sets of results (and confusion) ☹

Btw, Program-Created Indicator Predictors

- Designate a predictor as "**categorical**" in program syntax
  - Use CLASS in SAS; BY in SPSS; i. prefix in STATA; factor variable in R
- For a predictor with C categories, the program automatically then creates C new dummy codes, for example "group" with $C = 4$:

New Predictors Created Internally Mean This:	**Control**	**Treat1**	**Treat2**	**Treat3**
IsControl	**1**	0	0	0
IsTreat1	0	**1**	0	0
IsTreat2	0	0	**1**	0
IsTreat3	0	0	0	**1**

- It then figures out how many of these internal predictor variables are needed—if using an intercept (the default), then it's $C - 1$, not all C
- It enters them until it hits that criterion—if it leaves the last one out (as when you have an intercept), then last category becomes your reference
- Everywhere in syntax you refer to the categorical predictor, you must tell the program what to do with EACH of these internal predictor variables

What about Semi-Continuous Predictors?

- Some predictors contain both "kinds" and "amount" info
  - "Kinds" → mixtures of populations
  - "Amount" → severity within some (nested effect within subpopulation)
- Solution: an "if and how much" coding scheme, as shown →
  - "piecewise slopes" or "linear splines"

# Daily Cigarettes	smoker: 0=no, 1=yes	smkamt: If smoker, how much >1?
0	0	0
1	1	0
2	1	1
3	1	2
4	1	3
5	1	4
6	1	5

R:

```
data$smoker = NA # Make 2 empty vars
data$smkamt = NA
data$smoker[which(data$cig==0)]=0
data$smkamt[which(data$cig==0)]=0
data$smoker[which(data$cig>0)]=1
data$smkamt[which(data$cig>0)]=
   data$cig[which(data$cig>0)]-1
```

STATA:

```
gen smoker=. // Make 2 empty vars
gen smkamt=.
replace smoker=0      if cig==0
replace smkamt=0      if cig==0
replace smoker=1      if cig>0
replace smkamt=cig-1 if cig>0
```

SAS:

```
* Do not really need 2 empty vars first;
smoker=.; smkamt=.;
     IF cig=0 THEN DO; smoker=0; smkamt=0;     END;
ELSE IF cig>0 THEN DO; smoker=1; smkamt=cig-1; END;
```

How Many Fixed Slopes Per Predictor?

- "**Linear**" in GLM refers to "slope*variable + slope*variable" format
  - This means the x_i predictors can also be nonlinear terms (like x_i^2 to create a curve for x_i), which is then called "**nonlinear in the variables**"
  - The alternative, "**nonlinear in the parameters**" would have a nonlinear form, e.g., this exponential model: $\hat{\boldsymbol{y}}_{\boldsymbol{i}} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}[\boldsymbol{exp}(\boldsymbol{\beta_2}(\boldsymbol{x1_i}))]$

- The **role of each predictor $\boldsymbol{x_i}$** in creating a custom expected outcome $\boldsymbol{y_i}$ can be described through one or more fixed slopes
  - **One slope** is sufficient to capture the mean difference between two categories for a **binary $\boldsymbol{x_i}$** or to capture a **linear effect** of a quantitative x_i (or exponential for log x_i or logistic for logit x_i)
  - **More than one slope** may be needed to capture other nonlinear effects of a quantitative x_i (e.g., **quadratic** or **piecewise** trends)
  - $\boldsymbol{C-1}$ **slopes** are needed to capture the mean differences in the outcome across a **categorical predictor** with $\boldsymbol{C}$ **categories**
  - When **multiple slopes** are needed to describe the effect of a predictor, you will likely want a **joint hypothesis test** for all of them together…

Multivariate Wald Tests of Fixed Effects

- General test for significance of **multiple fixed effects** at once (can be requested via SAS CONTRAST, STATA TEST; code differs by package in R)—you have likely already seen these special cases…

- GLM: Whether a set of fixed slopes significantly explains y_i variance (i.e., if $R^2 > 0$) is tested via "**Multivariate Wald Test"** or $\boldsymbol{F}$**-test**"
  - $F(DF_{num}, DF_{den}) = \frac{SS_{model}/(k-1)}{SS_{residual}/(N-k)} = \frac{(N-k)R^2}{(k-1)(1-R^2)} = \frac{weighted\ \mathbf{known}\ info}{weighted\ \mathbf{unknown}\ info}$
  - $\boldsymbol{F}$**-test** evaluates model R^2 *per DF spent to get it and DF leftover*
  - $\boldsymbol{R^2} = \frac{SS_{total} - SS_{residual}}{SS_{total}}$ = square of $\boldsymbol{r}$ between predicted $\boldsymbol{\hat{y}_i}$ and $\boldsymbol{y_i}$

- e.g., "Omnibus" F-test for the slopes of the main effect of a variable with $C > 2$ categories (or for its interaction with other predictors)
- e.g., Model R^2 change F-test in hierarchical regression (for grouping sets of predictors together and testing their joint contribution)
- Btw, without denominator DF, $\boldsymbol{F}$ is replaced by $\boldsymbol{\chi^2}$ $(= F * DF_{num})$
- Btw, when testing only 1 slope, $t^2 = F$ and $z^2 = \chi^2$

A Taxonomy of Fixed Effect Interpretations

- In the most common statistical models, **fixed effects will be either**:
  - An **intercept** that provides an expected (conditional) y_i outcome,
  - Or **a slope** for the difference in y_i per unit difference in x_i predictor
    - Slopes for quantitative and categorical predictors are treated the same

- **All slopes** can be described as falling within one of three categories: ***bivariate marginal***, ***unique marginal***, or ***unique conditional***
  - In models with only **one fixed slope**, that slope's main effect is ***bivariate marginal*** (is uncontrolled and applies across all persons)
  - In models with **more than one fixed slope**, each slope's main effect is ***unique*** (it controls for the overlap in contribution with each other slope)
    - If a predictor is not part of an interaction term, its ***unique effect is marginal*** (it controls for the other slopes, but its effect still applies across all persons)
    - If a predictor is part of one or more interaction terms, its ***unique effect is conditional,*** which means it is **specific to each interacting predictor = 0**
      - **Unique conditional** effects are also called **"simple main effects" (simple slopes)**

Fixed Slope Interpretations: Example

- Model: $y_i = \beta_0 + \beta_1(w_i) + e_i$
  - β_1 is "bivariate marginal": difference in y_i per unit w_i (uncontrolled)

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + e_i$
  - β_1 is "<u>unique</u> marginal": diff in y_i per unit w_i, <u>controlling</u> for x_i and z_i
  - β_2 is "<u>unique</u> marginal": diff in y_i per unit x_i, <u>controlling</u> for w_i and z_i
  - β_3 is "<u>unique</u> marginal": diff in y_i per unit z_i, <u>controlling</u> for w_i and x_i

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + e_i$
  - β_1 is "unique marginal": diff in y_i per unit w_i, controlling for x_i and z_i
  - β_2 is "unique <u>conditional</u>": diff in y_i per unit x_i, controlling for w_i and z_i, <u>specifically when $z_i = 0$</u> (i.e., <u>β_2</u> is a "simple" main effect slope)
  - β_3 is "unique <u>conditional</u>": diff in y_i per unit z_i, controlling for w_i and x_i, <u>specifically when $x_i = 0$</u> (i.e., <u>β_3</u> is a "simple" main effect slope)
  - β_4 is "unique marginal" (unconditional), but how do we interpret it???

Interpreting Interaction Terms

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + e_i$
  - β_4 is "unique marginal" → interaction slope controlling for other slopes
  - Rather than talk about what happens to the predicted outcome y_i, interaction slopes are described by **what they do to their main effects**

- **Two-way** interaction has **two equally correct** interpretations:
  - How slope of x_i is moderated by z_i: β_4 = difference in β_2 per unit z_i
  - How slope of z_i is moderated by x_i: β_4 = difference in β_3 per unit x_i

- So model-implied effects of x_i and z_i are **linear combinations** (find common terms, factor out predictor the slope is for, and then the term in brackets is model-implied predictor effect)
  - Model-implied effect of x_i: $\beta_2(x_i) + \beta_4(x_i)(z_i)$ → $[\beta_2 + \beta_4(z_i)](x_i)$
  - Model-implied effect of z_i: $\beta_3(z_i) + \beta_4(x_i)(z_i)$ → $[\beta_3 + \beta_4(x_i)](z_i)$
  - Result can be found using SAS ESTIMATE, STATA LINCOM, or R GLHT
  - Many of our examples this semester will have interaction terms!

Only 4 Kinds of Interactions

- There are only 4 kinds of interactions: they make each of their main effect slopes (more/less) (positive/negative)
  - **More** positive or more negative → effect becomes **stronger**, known as "over-additive" interaction
  - **Less** positive or less negative → effect becomes **weaker**, known as "under-additive" interaction

- Model: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{w_i}) + \boldsymbol{\beta_2}(\boldsymbol{x_i}) + \boldsymbol{\beta_3}(\boldsymbol{z_i}) + \boldsymbol{\beta_4}(\boldsymbol{x_i})(\boldsymbol{z_i}) + \boldsymbol{e_i}$

Slope of x_i is β_2=	**Interaction Slope is β_4=**	**So the effect of x_i is ??? per unit higher z_i**
10	2	more positive (by $\boldsymbol{\beta_4}$)
10	-2	less positive (by $\boldsymbol{\beta_4}$)
-10	-2	more negative (by $\boldsymbol{\beta_4}$)
-10	2	less negative (by $\boldsymbol{\beta_4}$)

When There's More than One Interaction

- Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i)}$
  $\boldsymbol{+ \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + e_i}$

- Now all main effect slopes are "unique conditional" (simple):
  - $\boldsymbol{\beta_1}$ = diff in $\boldsymbol{y_i}$ per unit $\boldsymbol{w_i}$ specifically when $\boldsymbol{z_i = 0}$
  - $\boldsymbol{\beta_2}$ = diff in $\boldsymbol{y_i}$ per unit $\boldsymbol{x_i}$ specifically when $\boldsymbol{z_i = 0}$
  - $\boldsymbol{\beta_3}$ = diff in $\boldsymbol{y_i}$ per unit $\boldsymbol{z_i}$ specifically when $\boldsymbol{w_i = 0}$ **and** $\boldsymbol{x_i = 0}$

- Interaction slopes ($\boldsymbol{\beta_4}$ and $\boldsymbol{\beta_5}$) are "unique marginal"
  - $\boldsymbol{\beta_4}$ is now controlling for $\boldsymbol{\beta_5}$, but interpretation of $\boldsymbol{\beta_4}$ is unchanged:
    How slope of $\boldsymbol{x_i}$ is moderated by $\boldsymbol{z_i}$: $\boldsymbol{\beta_4}$ = difference in $\boldsymbol{\beta_2}$ per unit $\boldsymbol{z_i}$
    How slope of $\boldsymbol{z_i}$ is moderated by $\boldsymbol{x_i}$: $\boldsymbol{\beta_4}$ = difference in $\boldsymbol{\beta_3}$ per unit $\boldsymbol{x_i}$
  - New $\boldsymbol{\beta_5}$ has two equally correct interpretations:
    How slope of $\boldsymbol{w_i}$ is moderated by $\boldsymbol{z_i}$: $\boldsymbol{\beta_5}$ = difference in $\boldsymbol{\beta_1}$ per unit $\boldsymbol{z_i}$
    How slope of $\boldsymbol{z_i}$ is moderated by $\boldsymbol{w_i}$: $\boldsymbol{\beta_5}$ = difference in $\boldsymbol{\beta_3}$ per unit $\boldsymbol{w_i}$

When There's More than One Interaction

- Model: $\boldsymbol{y_i} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}(\boldsymbol{w_i}) + \boldsymbol{\beta_2}(\boldsymbol{x_i}) + \boldsymbol{\beta_3}(\boldsymbol{z_i})$
$+ \boldsymbol{\beta_4}(\boldsymbol{x_i})(\boldsymbol{z_i}) + \boldsymbol{\beta_5}(\boldsymbol{w_i})(\boldsymbol{z_i}) + \boldsymbol{e_i}$

- Model-implied effects of $\boldsymbol{w_i}$, $\boldsymbol{x_i}$ and $\boldsymbol{z_i}$ are **linear combinations** (find common terms, factor out predictor the slope is for, and then the term in brackets is the equation for the simple effect)
  - Effect of $\boldsymbol{w_i}$: $\boldsymbol{\beta_1}(\boldsymbol{w_i}) + \boldsymbol{\beta_5}(\boldsymbol{w_i})(\boldsymbol{z_i}) \rightarrow [\boldsymbol{\beta_1} + \boldsymbol{\beta_5}(\boldsymbol{z_i})](\boldsymbol{w_i})$
  - Effect of $\boldsymbol{x_i}$: $\boldsymbol{\beta_2}(\boldsymbol{x_i}) + \boldsymbol{\beta_4}(\boldsymbol{x_i})(\boldsymbol{z_i}) \rightarrow [\boldsymbol{\beta_2} + \boldsymbol{\beta_4}(\boldsymbol{z_i})](\boldsymbol{x_i})$
  - Effect of $\boldsymbol{z_i}$: $\boldsymbol{\beta_3}(\boldsymbol{z_i}) + \boldsymbol{\beta_4}(\boldsymbol{x_i})(\boldsymbol{z_i}) + \boldsymbol{\beta_5}(\boldsymbol{w_i})(\boldsymbol{z_i}) \rightarrow [\boldsymbol{\beta_3} + \boldsymbol{\beta_4}(\boldsymbol{x_i}) + \boldsymbol{\beta_5}(\boldsymbol{w_i})](\boldsymbol{z_i})$

- For quantitative moderators, **regions of significance** (see Hoffman 2015 ch. 2; Finsaas & Goldstein, 2021) can identify **moderator boundary values** for direction and significance of main effect slope
  - e.g., at what values of moderator $\boldsymbol{z_i}$ does the effect of $\boldsymbol{w_i}$ go from:
    (a) significantly negative to nonsignificant?
    (b) nonsignificant to significantly positive?

Interactions Involving Categorical Predictors

- When using manual contrasts for predictors with 3 or more categories, **interactions must be specified with separate dummy-coded predictors**
- If the program creates the dummy-coded predictors for you, all needed interaction predictors will be automatically included (but be careful!)

- e.g., **Adding an interaction of 4-category "group" with age (0=85):**
  - New predictors we must create for the model:
    d1 = 0, 1, 0, 0 → difference between Control and Treat1
    d2 = 0, 0, 1, 0 → difference between Control and Treat2
    d3 = 0, 0, 0, 1 → difference between Control and Treat3

$$y_i = \beta_0 + \beta_1(d1_i) + \beta_2(d2_i) + \beta_3(d3_i) + \beta_4(Age_i - 85) + \beta_5(d1_i)(Age_i - 85) + \beta_6(d2_i)(Age_i - 85) + \beta_7(d3_i)(Age_i - 85) + e_i$$

- Multivariate Wald test would be needed to lump together the interaction contrasts (β_5, β_6, and β_7) in order to test the "group*age" interaction
- Group difference slopes (β_1, β_2, and β_3) are each conditional on age = 85
- Age slope (β_4) is specific to the control group (when interactions = 0)
- But the model provides age slopes for each group, as well as group differences at any age as linear combinations of the fixed effects…

Interactions Involving Categorical Predictors

- **Adding an interaction of 4-category "group" with age (0=85):**
  - New predictors we must create for the model:
    - ***d1*** = 0, 1, 0, 0 → difference between Control and Treat1
    - ***d2*** = 0, 0, 1, 0 → difference between Control and Treat2
    - ***d3*** = 0, 0, 0, 1 → difference between Control and Treat3

$$y_i = \beta_0 + \beta_1(d1_i) + \beta_2(d2_i) + \beta_3(d3_i) + \beta_4(Age_i - 85) + \beta_5(d1_i)(Age_i - 85) + \beta_6(d2_i)(Age_i - 85) + \beta_7(d3_i)(Age_i - 85) + e_i$$

- **Equations for model-implied effects: [slope] (predictor)**
  - Effect of Age in Control group: $[\beta_4 + \beta_5(0) + \beta_6(0) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat1 group: $[\beta_4 + \beta_5(1) + \beta_6(0) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat2 group: $[\beta_4 + \beta_5(0) + \beta_6(1) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat3 group: $[\beta_4 + \beta_5(0) + \beta_6(0) + \beta_7(1)](Age_i - 85)$
  - Control vs. Treat1 for any age: $[\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - Control vs. Treat2 for any age: $[\beta_2 + \beta_6(Age_i - 85)](d2_i)$
  - Control vs. Treat3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i)$
  - T1 vs T2 for any age: $[\beta_2 + \beta_6(Age_i - 85)](d2_i) - [\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - T1 vs T3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i) - [\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - T2 vs T3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i) - [\beta_2 + \beta_6(Age_i - 85)](d2_i)$

What about 3-way interactions???

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i)$
$+ \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + \beta_6(x_i)(w_i)$
$+ \beta_7(w_i)(x_i)(z_i) + e_i$

- **Simple main effects make the predicted outcome higher or lower**
  - 1 possible interpretation for each simple main effect slope
  - Each simple main effect is conditional on other interacting predictors = 0
- **Each 2-way interaction (3 of them in a 3-way model) makes its simple main effect slopes (more/less) (positive/negative)**
  - So there are 2 possible interpretations for each 2-way interaction
  - Each simple 2-way interaction is conditional on third predictor = 0
- **The 3-way interaction makes each of its *2-way simple interaction slopes* (more/less) (positive/negative)**
  - So there are 3 possible interpretations of the 3-way interaction
  - Is highest-order term in model, so is unconditional (marginal)

3-way Interactions Follow the Same Rules

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i)$
$$+ \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + \beta_6(x_i)(w_i)$$
$$+ \beta_7(w_i)(x_i)(z_i) + e_i$$

- **Model-implied simple (conditional) main effects:**
  - Effect of w_i: $[\beta_1 + \beta_5(z_i) + \beta_6(x_i) + \beta_7(x_i)(z_i)](w_i)$
  - Effect of x_i: $[\beta_2 + \beta_4(z_i) + \beta_6(w_i) + \beta_7(w_i)(z_i)](x_i)$
  - Effect of z_i: $[\beta_3 + \beta_4(x_i) + \beta_5(w_i) + \beta_7(w_i)(x_i)](z_i)$

- **Model-implied simple (conditional) 2-way interactions:**
  - Effect of x_i by z_i: $[\beta_4 + \beta_7(w_i)](x_i)(z_i)$
  - Effect of w_i by z_i: $[\beta_5 + \beta_7(x_i)](w_i)(z_i)$
  - Effect of x_i by w_i: $[\beta_6 + \beta_7(z_i)](x_i)(w_i)$

Interpreting Interactions: Summary

- Interactions represent "moderation" – the idea that the effect of one predictor depends upon the level of the other(s)

- The main effect slopes WILL CHANGE once their predictors are part of an interaction, because they now mean different things:
  - Main effect → Simple effect specifically when interacting predictor(s) = 0
  - Need to have 0 as a meaningful value for each predictor for that reason

- Rules for interpreting conditional (simple) fixed slopes:
  - Intercepts are conditional on (i.e., get adjusted by) main effect slopes
  - Main effects are conditional on two-way interactions
  - Two-way interactions are conditional on three-way interactions
  - Highest-order term is unconditional → same regardless of centering

Bonus: Dummy vs. Effect Coding

TABLE 10.3. Four Ways of Coding Age Cohort and the Group Means Defined in Terms of the Regression Coefficients and Regression Constant

Age cohort by increasing age	D_1	D_2	D_3	Mean of Y
Indicator coding				
Generation Y	1	0	0	$\overline{Y}_1 = b_0 + b_1$
Generation X	0	1	0	$\overline{Y}_2 = b_0 + b_2$
Baby boomer	0	0	1	$\overline{Y}_3 = b_0 + b_3$
Pre-baby boomer	0	0	0	$\overline{Y}_4 = b_0$
Sequential coding				
Generation Y	0	0	0	$\overline{Y}_1 = b_0$
Generation X	1	0	0	$\overline{Y}_2 = b_0 + b_1$
Baby boomer	1	1	0	$\overline{Y}_3 = b_0 + b_1 + b_2$
Pre-baby boomer	1	1	1	$\overline{Y}_4 = b_0 + b_1 + b_2 + b_3$
Helmert coding				
Generation Y	−3/4	0	0	$\overline{Y}_1 = b_0 - \frac{3}{4}b_1$
Generation X	1/4	−2/3	0	$\overline{Y}_2 = b_0 + \frac{1}{4}b_1 - \frac{2}{3}b_2$
Baby boomer	1/4	1/3	−1/2	$\overline{Y}_3 = b_0 + \frac{1}{4}b_1 + \frac{1}{3}b_2 - \frac{1}{2}b_3$
Pre-baby boomer	1/4	1/3	1/2	$\overline{Y}_4 = b_0 + \frac{1}{4}b_1 + \frac{1}{3}b_2 + \frac{1}{2}b_3$
Effect coding				
Generation Y	1	0	0	$\overline{Y}_1 = b_0 + b_1$
Generation X	0	1	0	$\overline{Y}_2 = b_0 + b_2$
Baby boomer	0	0	1	$\overline{Y}_3 = b_0 + b_3$
Pre-baby boomer	−1	−1	−1	$\overline{Y}_4 = b_0 - b_1 - b_2 - b_3$

- **Indicator** and **sequential** coding each use one designated category as the reference
- **Helmert coding** "quantifies the difference in means between one group and the mean of the means in all higher-coded groups"
- **Effect coding** uses the grand mean across (equally weighted categories) as the reference; slopes give mean differences relative to grand mean

Table 10.3 on p. 278 of: Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford.