General Linear Models with Interactions: Testing Moderation!

- Topics—a subset of EDF 9770 Lecture 5 for review in EDF 9850:
  - Slopes of predictors within interactions: from unique main (marginal) effects to unique simple (conditional) effects
    - The 4 possible kinds of interactions
    - Model-implied slopes as linear combinations of model slopes
    - Interactions with categorical predictors

GLM with an Interaction: $y_i = \beta_0 + \beta_1(x_i) + \beta_2(z_i) + \boxed{\beta_3(x_i)(z_i)} + e_i$

- **Interaction slopes (β_3 here) test "Moderation"**: whether a predictor's slope depends on the value of another predictor
  - Either predictor can be "the moderator" (is interpretive distinction only)

- **Interactions can always be tested** for any combination of categorical or quantitative predictors, although traditionally…
  - **In "ANOVA"**: By default, includes all possible interactions of categorical predictors
    - Oddly, nonsignificant interactions are usually kept in the model
  - **In "ANCOVA"**: Quantitative predictors ("covariates") are not included in interaction terms → this is the "homogeneity of regression assumption"
    - But you don't have to assume this—it is always a testable hypothesis!
  - **In "Regression"**: No default—effects of predictors are as you specify
    - Requires most thought, but also allows the most flexibility

Main Effects of Predictors within Interactions

- "**Main effect**" **slopes** of predictors that are included in interaction terms should always remain in the model regardless of their significance
  - e.g., given $\boldsymbol{\beta_3(x_i)(z_i)}$, you must keep $\boldsymbol{\beta_1(x_i)}$ and $\boldsymbol{\beta_2(z_i)}$ in the model, too
  - Why? Because an interaction term creates an over-additive (enhancing) or under-additive (dampening) effect, so what it is additive to must be included for the interaction to correctly represent an "interaction"

- **Role of a two-way interaction is to adjust the** "**main effect**" **slopes** of the two predictors involved
  - But the idea of a "**marginal**" main effect slope (that holds for everyone) **no longer applies**: The main effect slopes become "**simple**" main effect slopes that are **conditional on each interacting predictor = 0**

- Note that this is a **different type of conditionality** than just "holding the other predictors constant" (which means constant at any value)
  - Simple main effect slopes are **held constant** (conditional on) the **0 value** of the interacting predictor(s)—these slopes would be different if 0 were defined differently by centering the interacting predictor elsewhere
  - This language can be confusing, so next is a taxonomy that may help…

A Taxonomy of Fixed Effect Interpretations

- In most common linear models, fixed effects will be either:
  - an **intercept** that provides an expected (conditional) y_i outcome,
  - or a **slope** for the difference in y_i per unit difference in x_i predictor

- **All slopes** can be described as falling within one of three categories: ***bivariate marginal***, ***unique marginal***, or ***unique conditional***
  - In models with only **one fixed slope**, that slope's main effect is ***bivariate marginal*** (is uncontrolled and applies across all persons)
  - In models with **more than one fixed slope**, each slope's main effect is ***unique*** (it controls for the overlap in contribution with each other slope)
    - If a predictor is not part of an interaction term, its ***unique effect is marginal*** (it controls for the other slopes, but its effect still applies across all persons)
    - **NEW** If a predictor is part of one or more interaction terms, its ***unique effect is conditional*,** which means it is **specific to each interacting predictor = 0**
      - **Unique conditional** effects are also called **"simple main effects" (simple slopes)**

Practice Labeling Fixed Slopes—Choices: *bivariate marginal, unique marginal, or unique conditional*

Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + e_i}$

- Label for $\boldsymbol{\beta_1}$ slope of $\boldsymbol{w_i}$ = *bivariate marginal*

Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + e_i}$

- Label for $\boldsymbol{\beta_1}$ slope of $\boldsymbol{w_i}$ = *unique marginal*
- Label for $\boldsymbol{\beta_2}$ slope of $\boldsymbol{x_i}$ = *unique conditional on z =0*
- Label for $\boldsymbol{\beta_3}$ slope of $\boldsymbol{z_i}$ = *unique conditional on x =0*
- Label for $\boldsymbol{\beta_4}$ slope of $\boldsymbol{x_i z_i}$ interaction term = *unique marginal*

The 4 Possible Kinds of Interactions

- There are only 4 kinds of interactions: they make each of their main effect slopes more/less positive/negative
  - **More** positive or more negative → effect becomes **stronger**, known as "over-additive" interaction
  - **Less** positive or less negative → effect becomes **weaker**, known as "under-additive" interaction

- Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + e_i}$

Slope of x_i is β_2=	**Interaction Slope is β_4=**	**So β_4 makes effect of x_i ??? per unit higher z_i**
10	2	More positive
-10	2	Less negative
10	-2	Less positive
-10	-2	More negative

Fixed Effects: Why Centering Matters

- y_i = Student achievement (GPA as percentage out of 100)
  x_i = Parent **attitudes** about education (measured on 1–5 scale)
  z_i = Parent **education** level (measured in years of education)

$$GPA_i = \beta_0 + \beta_1(Att_i) + \beta_2(Ed_i) + \beta_3(Att_i)(Ed_i) + e_i$$
$$GPA_i = 30 + 1(Att_i) + 2(Ed_i) + 0.5(Att_i)(Ed_i) + e_i$$

- **Interpret β_0: predicted GPA when att=0 and ed=0**
- **Interpret β_1: change in GPA per unit att *when ed=0***
- **Interpret β_2: change in GPA per unit ed *when att=0***
- **Interpret β_3: Attitude as Moderator: change in ed slope per unit att**
  Education as Moderator: change in att slope per unit ed

- **Predicted GPA** for **Attitude = 3** and **Ed = 12**?
  75 = 30 + 1*(3) + 2*(12) + 0.5*(3)*(12)

How Centering Changes the Fixed Effects

- $\boldsymbol{y_i}$ = Student achievement (GPA as percentage out of 100)
  $\boldsymbol{x_i}$ = Parent **attitudes** about education (now centered at **3**)
  $\boldsymbol{z_i}$ = Parent years of **education** (now centered at **12**)

$$GPA_i = \beta_0 + \beta_1(Att_i - 3) + \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12) + e_i$$
$$GPA_i = 75 + 7(Att_i - 3) + 3.5(Ed_i - 12) + 0.5(Att_i - 3)(Ed_i - 12) + e_i$$

- **Interpret β_0: predicted GPA for att=3 and ed=12**
- **Interpret β_1: change in GPA per unit att *when ed=12***
- **Interpret β_2: change in GPA per unit ed *when att=3***
- **Interpret β_3: Attitude as Moderator: change in ed slope per unit att**

  Education as Moderator: change in att slope per unit ed

- But how did I know what the new fixed effects would be???

Model-Implied Predicted Outcomes

- **Predicted outcomes = expected outcomes = conditional means**
  - ALL model effects must be included (or else are assumed = 0)

$$\widehat{GPA_i} = \beta_0 + \beta_1(Att_i - 3) + \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12)$$

R: Values are multipliers in glht *in order of the fixed effects output*:

```
glhtName = glht(model=ModelName, linfct=rbind(
"Yhat: Att=5 Ed=16" c(1, 2, 4, 8),
"Yhat: Att=1 Ed=9"  c(1,-2,-3, 6),
"Yhat: Att=3 Ed=20" c(1, 0, 8, 0)))
```

Model-Implied Predictor Simple Slopes

- Example equation for predicted GPA using centered predictors:
  $\widehat{GPA_i} = \beta_0 + \beta_1(Att_i - 3) + \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12)$

- This model equation provides predictions for:
  - Expected outcome given any combination of predictor values
  - Any conditional (simple) main effect slopes implied by interaction term
  - **Any slope can be found as: what it is + what *modifies* it**

- Three steps to get any model-implied simple main effect slope:

1. **Identify** all terms in model involving the predictor of interest
2. **Factor out** common predictor variable to find slope linear combination
3. **Compute** estimate and SE for slope linear combination
   - *By "compute" of course I mean "ask the program to do this for you"*

Model-Implied Predictor Simple Slopes

- Example equation for predicted GPA using centered predictors:
  $\widehat{GPA_i} = \beta_0 + \beta_1(Att_i - 3) + \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12)$

1. **Identify** all slopes in model involving the predictor of interest

   To get attitudes slope: $Est = \beta_1(Att_i - 3) + \beta_3(Att_i - 3)(Ed_i - 12)$

   To get education slope: $Est = \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12)$

2. **Factor out** predictor of interest to find the slope's linear combination

   To get attitudes slope: $Est = [\beta_1 + \beta_3 (Ed_i - 12)]$ **that will multiply** $(Att_i - 3)$

   To get education slope: $Est = [\beta_2 + \beta_3(Att_i - 3)]$ **that will multiply** $(Ed_i - 12)$

- Btw, the SEs for the new slopes provided by the program come from:
  - SE^2 = sampling variance of slope estimate → e.g., $Var(\beta_1) = SE^2_{\beta_1}$

  attitudes slope: $SE^2 = Var(\beta_1) + Var(\beta_3)(Ed_i - 12) + 2Cov(\beta_1, \beta_3)(Ed_i - 12)$

  education slope: $SE^2 = Var(\beta_2) + Var(\beta_3)(Att_i - 3) + 2Cov(\beta_2, \beta_3)(Att_i - 3)$

Model-Implied Predictor Simple Slopes

- **To request <u>predicted simple slopes</u> (= simple main effects):**
  - **Include ONLY** the fixed effects that contain the predictor of interest

$$\widehat{GPA_i} = \beta_0 + \beta_1(Att_i - 3) + \beta_2(Ed_i - 12) + \beta_3(Att_i - 3)(Ed_i - 12)$$

→ attitudes slope: $Est = [\beta_1 + \beta_3(Ed_i - 12)]$ that multiplies $(Att_i - 3)$

→ education slope: $Est = [\beta_2 + \beta_3(Att_i - 3)]$ that multiplies $(Ed_i - 12)$

R: Values are multipliers in GLHT in order of fixed effects:

```
glhtName = glht(model=ModelName, linfct=rbind(
"Att Slope if Ed=10" c(0,_1_,_0_,_-2_),
"Att Slope if Ed=18" c(0,_1_,_0_,_ 6_),
"Ed Slope  if Att=2" c(0,_0_,_1_,_-1_),
"Ed Slope  if Att=5" c(0,_0_,_1_,_ 2_)))
```

When There's More than One Interaction

- Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + e_i}$

- Now all main effect slopes are "unique conditional" (simple):
  - $\boldsymbol{\beta_1}$ = diff in $\boldsymbol{y_i}$ per one-unit $\boldsymbol{w_i}$ specifically when $\boldsymbol{z_i = 0}$
  - $\boldsymbol{\beta_2}$ = diff in $\boldsymbol{y_i}$ per one-unit $\boldsymbol{x_i}$ specifically when $\boldsymbol{z_i = 0}$
  - $\boldsymbol{\beta_3}$ = diff in $\boldsymbol{y_i}$ per one-unit $\boldsymbol{z_i}$ specifically when $\boldsymbol{w_i = 0}$ **and** $\boldsymbol{x_i = 0}$

- Interaction slopes ($\boldsymbol{\beta_4}$ and $\boldsymbol{\beta_5}$) are "unique marginal"
  - $\boldsymbol{\beta_4}$ is now controlling for $\boldsymbol{\beta_5}$, but interpretation of $\boldsymbol{\beta_4}$ is unchanged:
    How slope of $\boldsymbol{x_i}$ is moderated by $\boldsymbol{z_i}$: $\boldsymbol{\beta_4}$ = diff in $\boldsymbol{\beta_2}$ per one-unit $\boldsymbol{z_i}$
    How slope of $\boldsymbol{z_i}$ is moderated by $\boldsymbol{x_i}$: $\boldsymbol{\beta_4}$ = diff in $\boldsymbol{\beta_3}$ per one-unit $\boldsymbol{x_i}$
  - New $\boldsymbol{\beta_5}$ has two equally correct interpretations:
    How slope of $\boldsymbol{w_i}$ is moderated by $\boldsymbol{z_i}$: $\boldsymbol{\beta_5}$ = diff in $\boldsymbol{\beta_1}$ per one-unit $\boldsymbol{z_i}$
    How slope of $\boldsymbol{z_i}$ is moderated by $\boldsymbol{w_i}$: $\boldsymbol{\beta_5}$ = diff in $\boldsymbol{\beta_3}$ per one-unit $\boldsymbol{w_i}$

When There's More than One Interaction

- Model: $\boldsymbol{y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + e_i}$

- Model-implied slopes of $\boldsymbol{w_i}$, $\boldsymbol{x_i}$ and $\boldsymbol{z_i}$ are **linear combinations:** (1) find common terms, (2) factor out the predictor the slope is for, and (3) then the term in brackets is model-implied predictor slope
  - Slope of $\boldsymbol{w_i}$: $\boldsymbol{\beta_1(w_i) + \beta_5(w_i)(z_i) \rightarrow [\beta_1 + \beta_5(z_i)](w_i)}$
  - Slope of $\boldsymbol{x_i}$: $\boldsymbol{\beta_2(x_i) + \beta_4(x_i)(z_i) \rightarrow [\beta_2 + \beta_4(z_i)](x_i)}$
  - Slope of $\boldsymbol{z_i}$: $\boldsymbol{\beta_3(z_i) + \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) \rightarrow [\beta_3 + \beta_4(x_i) + \beta_5(w_i)](z_i)}$

- **More than one interaction slope will be necessary for categorical predictors (i.e., ordinal or nominal predictors with 3+ groups)**
  - I will continue to show you the longer but more transparent way using binary-coded contrasts to represent group differences → matches model equation
  - An alternative is to let the program create the contrast for you factor variables in R
    - Can be more convenient but more prone to misinterpretation (so I'm not doing it here)

Interactions Involving Categorical Predictors

- When using manually-created binary contrasts for predictors with 3 or more categories, **interactions must be specified with ALL binary predictors** (done for you when using a factor variable)

- e.g., **Adding an interaction of 4-category "group" with age (0=85):**
  - New predictors we must create for the model:
    - ***d1*** = 0, 1, 0, 0 → difference between Control and Treat1
    - ***d2*** = 0, 0, 1, 0 → difference between Control and Treat2
    - ***d3*** = 0, 0, 0, 1 → difference between Control and Treat3

$$y_i = \beta_0 + \beta_1(d1_i) + \beta_2(d2_i) + \beta_3(d3_i) + \beta_4(Age_i - 85) + \beta_5(d1_i)(Age_i - 85) + \beta_6(d2_i)(Age_i - 85) + \beta_7(d3_i)(Age_i - 85) + e_i$$

- F-test needed to lump together the interaction slopes (β_5, β_6, and β_7) to test the "omnibus" (overall) group*age interaction
- Group difference slopes (β_1, β_2, and β_3) are "simple main effects" each conditional on age = 85
- Age slope (β_4) is a "simple main effect" specific to the control group (when interactions = 0)
- Model provides age slopes for each group, as well as group differences at any age as linear combinations…

Interactions Involving Categorical Predictors

- **Adding an interaction of 4-category "group" with age (0=85):**
  - New predictors we must create for the model:
    - ***d1*** = 0, 1, 0, 0 → difference between Control and Treat1
    - ***d2*** = 0, 0, 1, 0 → difference between Control and Treat2
    - ***d3*** = 0, 0, 0, 1 → difference between Control and Treat3

$$y_i = \beta_0 + \beta_1(d1_i) + \beta_2(d2_i) + \beta_3(d3_i) + \beta_4(Age_i - 85) + \beta_5(d1_i)(Age_i - 85) + \beta_6(d2_i)(Age_i - 85) + \beta_7(d3_i)(Age_i - 85) + e_i$$

- **Equations for model-implied effects: [slope] (predictor)**
  - Effect of Age in Control group: $[\beta_4 + \beta_5(0) + \beta_6(0) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat1 group: $[\beta_4 + \beta_5(1) + \beta_6(0) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat2 group: $[\beta_4 + \beta_5(0) + \beta_6(1) + \beta_7(0)](Age_i - 85)$
  - Effect of Age in Treat3 group: $[\beta_4 + \beta_5(0) + \beta_6(0) + \beta_7(1)](Age_i - 85)$
  - Control vs. Treat1 for any age: $[\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - Control vs. Treat2 for any age: $[\beta_2 + \beta_6(Age_i - 85)](d2_i)$
  - Control vs. Treat3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i)$
  - T1 vs T2 for any age: $[\beta_2 + \beta_6(Age_i - 85)](d2_i) - [\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - T1 vs T3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i) - [\beta_1 + \beta_5(Age_i - 85)](d1_i)$
  - T2 vs T3 for any age: $[\beta_3 + \beta_7(Age_i - 85)](d3_i) - [\beta_2 + \beta_6(Age_i - 85)](d2_i)$

What about 3-way interactions???

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + \beta_6(x_i)(w_i) + \beta_7(w_i)(x_i)(z_i) + e_i$

- **Simple main effects make the predicted outcome higher or lower**
  - 1 possible interpretation for each simple main effect slope
  - Each simple main effect is conditional on other interacting predictors = 0
- **Each 2-way interaction (3 of them in a 3-way model) makes its simple main effect slopes (more/less) (positive/negative)**
  - So there are 2 possible interpretations for each 2-way interaction
  - Each "simple" 2-way interaction is conditional on third predictor = 0
- **The 3-way interaction makes each of its *2-way simple interaction slopes* (more/less) (positive/negative)**
  - So there are 3 possible interpretations of a 3-way interaction!
  - Is highest-order term in model, so is unconditional (marginal)

3-Way Interactions Follow the Same Rules

- Model: $y_i = \beta_0 + \beta_1(w_i) + \beta_2(x_i) + \beta_3(z_i) + \beta_4(x_i)(z_i) + \beta_5(w_i)(z_i) + \beta_6(x_i)(w_i) + \beta_7(w_i)(x_i)(z_i) + e_i$

- **Model-implied simple (conditional) main effect slopes:**
  - Effect of w_i: $[\beta_1 + \beta_5(z_i) + \beta_6(x_i) + \beta_7(x_i)(z_i)](w_i)$
  - Effect of x_i: $[\beta_2 + \beta_4(z_i) + \beta_6(w_i) + \beta_7(w_i)(z_i)](x_i)$
  - Effect of z_i: $[\beta_3 + \beta_4(x_i) + \beta_5(w_i) + \beta_7(w_i)(x_i)](z_i)$

- **Model-implied simple (conditional) 2-way interactions:**
  - Effect of x_i by z_i: $[\beta_4 + \beta_7(w_i)](x_i)(z_i)$
  - Effect of w_i by z_i: $[\beta_5 + \beta_7(x_i)](w_i)(z_i)$
  - Effect of x_i by w_i: $[\beta_6 + \beta_7(z_i)](x_i)(w_i)$

Intermediate Summary

- Interactions create "**moderation**": the idea that the effect (slope) of one predictor **depends** upon the value of another predictor

- Predictors' **main effect slopes** will change once they are included in an interaction term, because **they now mean different things**:
  - Former "marginal main effect slopes" become "conditional (or simple) effect slopes" specifically when the interacting predictor = 0 (not just "holding constant")
  - Need to have **0 as a meaningful value** for each predictor for that reason

- **Rules for interpreting conditional (or simple) fixed slopes:**
  - Predicted outcomes are conditional on (get adjusted by) main effect slopes
    - Positive slopes create higher outcomes; negative slopes create lower outcomes
  - Main effect slopes are conditional (get adjusted by) on two-way interactions
    - Interactions make main effect slopes more/less positive or more/less negative
    - Btw, three-way interactions do the same thing to two-way interactions
  - Highest-order interaction slope is unconditional—it will stay the same regardless of centering