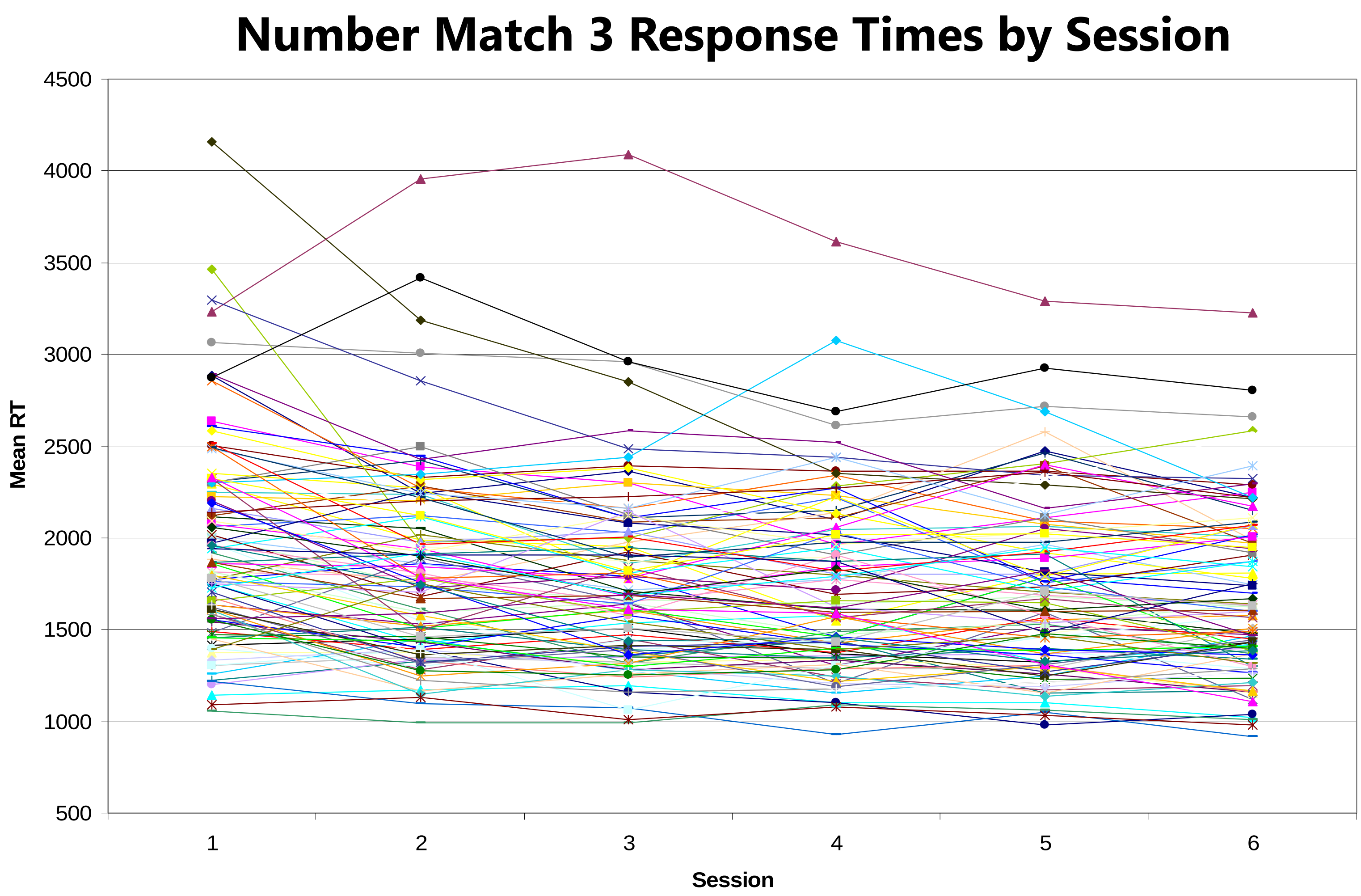
Describing Within-Person Change over Time

- Today's Class (for 3-4 days, actually):
  - The Big Picture of modeling change
  - Fixed and random effects models for nonlinear change:
    - Polynomial slopes
    - Piecewise slopes
    - Nonlinear change

Example Data Individual Observed Trajectories (N = 101, *n* = 6)



The Big Picture of Longitudinal Data: Model for the Means (Fixed Effects)

- What kind of change occurs on average over "time"?
  - What is the most appropriate **metric of time**?
    - Time in study (with predictors for BP differences in time)?
    - Time since birth (age)? Time to event (time since diagnosis)?
    - Measurement occasions need not be the same across persons or equally spaced (code time as exactly as possible)
  - What kind of **theoretical process** generated the observed trajectories, and thus what kind of model do we need?
    - Linear or nonlinear? Continuous or discontinuous? Does change keep happening or does it eventually stop?
    - Many options: polynomial, piecewise, and nonlinear families

The Big Picture of Longitudinal Data: Models for the Means (Fixed Effects)

- What kind of change occurs on average over "time"? Two baseline models for comparison:
  - "**Empty**" → only a fixed intercept (predicts no change)
  - "**Saturated**" → all occasion mean differences from time 0 (ANOVA model that uses # fixed effects= *n*) *** *may not be possible in unbalanced data*

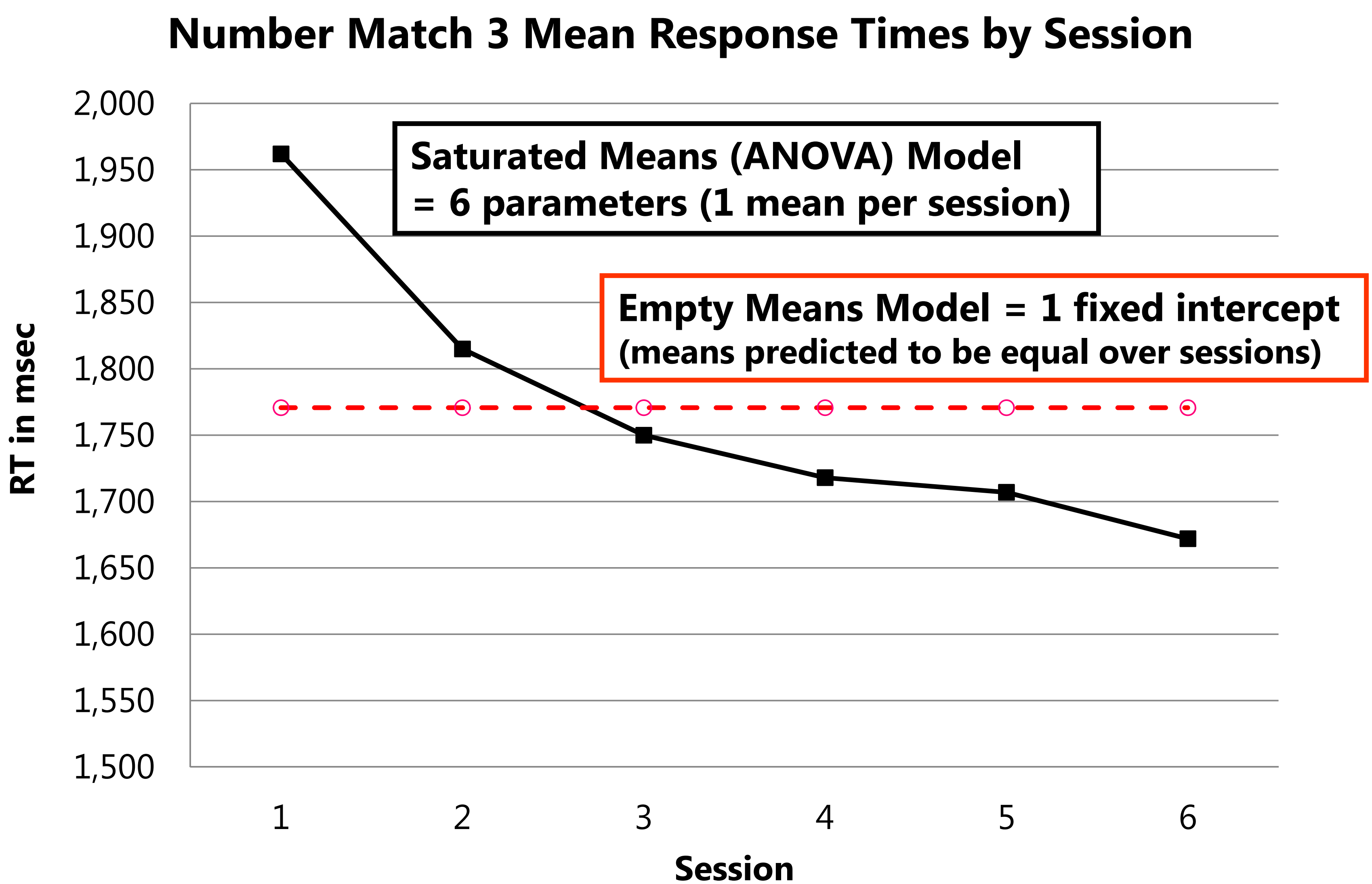
Parsimony ← **Empty Model:** Predicts NO change over time — **1 Fixed Effect**

In-between options: polynomial slopes, piecewise slopes, nonlinear models…

→ **Good fit** — **Saturated Means:** Reproduces mean at each occasion — **# Fixed Effects = # Occasions**

Name… that… Trajectory!

Baseline Models for the Means



The Big Picture of Longitudinal Data: Models for the Variance (Random Effects)

- *From a substantive perspective*: Are there **individual differences** in change?
  - Individual differences in the level of an outcome?
    - At what time point are individual differences in outcome level important for your hypotheses (beginning, middle, end)?
  - Individual differences in magnitude of change?
    - Each aspect of change (e.g., linear change, quadratic change) can potentially exhibit individual differences (data permitting)

- *From a statistical perspective*: What kind of pattern do the **variances and covariances** exhibit over time?
  - Do the variances increase or decrease over time?
  - Are the covariances differentially related based on time?

The Big Picture of Longitudinal Data: Models for the Variance

Compound Symmetry (CS)

$$\begin{bmatrix} \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e & \tau^2_{U_0} \\ \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0} & \tau^2_{U_0}+\sigma^2_e \end{bmatrix}$$

Most useful model: likely somewhere in between!

Unstructured (UN)

$$\begin{bmatrix} \sigma^2_{T1} & \sigma_{T12} & \sigma_{T13} & \sigma_{T14} \\ \sigma_{T21} & \sigma^2_{T2} & \sigma_{T23} & \sigma_{T24} \\ \sigma_{T31} & \sigma_{T32} & \sigma^2_{T3} & \sigma_{T43} \\ \sigma_{T41} & \sigma_{T42} & \sigma_{T43} & \sigma^2_{T4} \end{bmatrix}$$

Parsimony ⟵ ⟶ **Good fit**

Univariate RM ANOVA

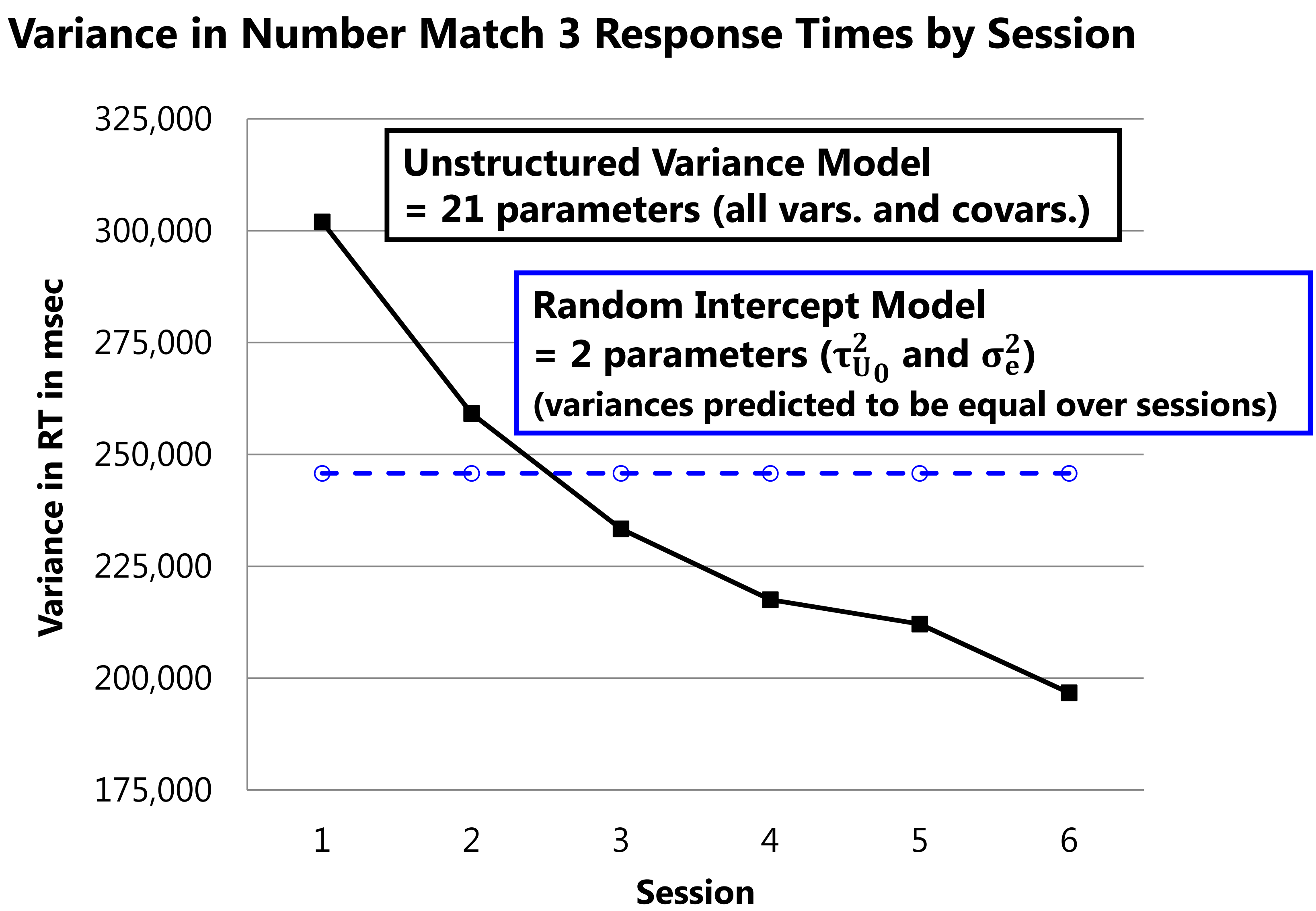
NAME ...THAT ... STRUCTURE!

Multivariate RM ANOVA

What is the pattern of variance and covariance over time?

CS and UN are just two of the many, many options available within MLM, including ***random effects models*** (for change) and ***alternative covariance structure models*** (for fluctuation).

Baseline Models for the Variance



Summary: Modeling Means and Variances

- We have two tasks in describing within-person change:

- **Choose a Model for the Means**
  - What kind of change in the outcome do we have on average?
  - What kind and how many **fixed effects** do we need to predict that mean change as parsimoniously but accurately as possible?

- **Choose a Model for the Variances**
  - What pattern do the variances and covariances of the outcome show over time because of individual differences in change?
  - What kind and how many **random effects** do we need to predict that pattern as parsimoniously but accurately as possible?

New Material: Absolute Fit in REML

- Answer key model (possible only for balanced data):
  - Means Model = Saturated Means
  - Variance Model = Unstructured R, or RI+UN($n-1$) equivalent

- Tests of absolute fit of any simpler means model against saturated means can only be done via $-2\Delta LL$ when using ML, but what if you need to use REML given small level-2 *N*?
  - Use a multivariate Wald test instead: add enough contrasts for occasion-specific mean differences to create saturated means, then test that group of contrasts (see example 6 for how to do so using CLASS/BY)

- Tests of absolute fit of any nested variance model against UN can be done using REML $-2\Delta LL$ if same means side (so keep the same fixed effects for time in each comparison model)

Name that trajectory… Polynomial?

- Predict **mean change** with **polynomial fixed effects of time**:
  - Linear → *constant* amount of change (up or down)
  - Quadratic → *change* in linear rate of change (acceleration/deceleration)
  - Cubic → *change* in acceleration/deceleration of linear rate of change (known in physics as jerk, surge, or jolt)
  - Terms work <u>together</u> to describe curved trajectories

- **Can have polynomial fixed time slopes UP TO: *n* – 1***
  - 3 occasions = 2nd order ($time^2$)= Fixed Quadratic Time or less
  - 4 occasions = 3rd order ($time^3$) = Fixed Cubic Time or less

- Interpretable polynomials past cubic are rarely seen in practice

*This rule can be broken in unbalanced data (but cautiously)

Interpreting Quadratic Fixed Effects

A Quadratic time effect is a two-way interaction: time*time

- Fixed quadratic time = "**half the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**

- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…

- The "twice" part comes from taking the derivatives of the function:

$$\text{Intercept (Position) at Time T: } \hat{y}_T = 50.0 + 4.0T + 0.3T^2$$

$$\text{First Derivative (Velocity) at Time T: } \frac{d\hat{y}_T}{d(T)} = 4.0 + 0.6T$$

$$\text{Second Derivative (Acceleration) at Time T: } \frac{d^2\hat{y}_T}{d(T)} = 0.6$$

Interpreting Quadratic Fixed Effects

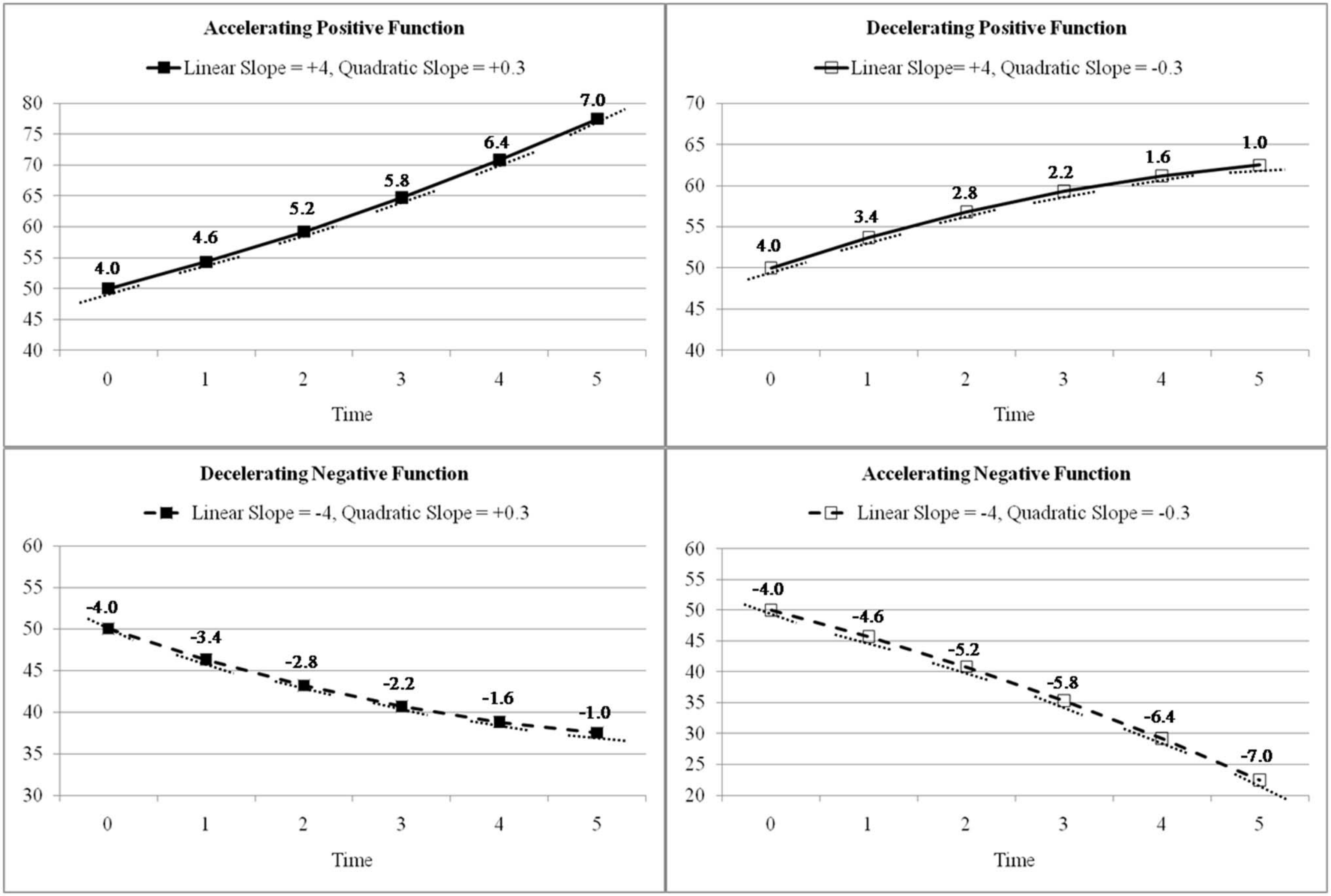
<u>A Quadratic time effect is a two-way interaction: time*time</u>

- Fixed quadratic = "**<u>half</u> the rate of acceleration/deceleration**"
- So to interpret it as how the linear time effect changes per unit time, **you must multiply the quadratic coefficient by 2**
- If fixed linear time slope = 4 at time 0, with quadratic slope = 0.3?
  - Instantaneous linear rate of Δ at time 0 = 4.0, at time 1 = 4.6…
- The "twice" part also comes from what you remember about the role of interactions with respect to their constituent main effects:

$$\hat{y} = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ$$
$$\text{Effect of } X = \beta_1 + \beta_3 Z$$
$$\text{Effect of } Z = \beta_2 + \beta_3 X$$
$$\hat{y}_T = \beta_0 + \beta_1 \text{Time}_T + \_\_\_\_ + \beta_3 \text{Time}_T^2$$
$$\text{Effect of Time}_T = \beta_1 + 2\beta_3 \text{Time}_T$$

- Because time is interacting with itself, there is no second main effect in the model for the interaction to modify as usual. So the quadratic time effect gets applied <u>twice</u> to the <u>one</u> (main) linear effect of time.

Examples of Fixed Quadratic Time Effects



Conditionality of Polynomial Fixed Time Effects

- We've seen how main effects become conditional simple effects once they are part of an interaction
- The same is true for polynomial **fixed effects of time**:
  - ***Fixed Intercept Only?***
    - Fixed Intercept = predicted mean of Y *for any occasion (= grand mean)*
  - ***Add Fixed Linear Time?***
    - Fixed Intercept = **now** predicted mean of Y from linear time *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Linear Time = mean linear rate of change *across all occasions* *(would be the same if time was centered elsewhere)*
  - ***Add Fixed Quadratic Time?***
    - Fixed Intercept = still predicted mean of Y *at time=0 (but from quadratic model)* (*would be different if time was centered elsewhere*)
    - Fixed Linear Time = **now** mean linear rate of change *at time=0* *(would be different if time was centered elsewhere)*
    - Fixed Quadratic Time = half the mean rate of acceleration or deceleration of change *across all occasions* (i.e., the linear slope changes the same over time)

Polynomial Fixed vs. Random Time Effects

- **Polynomial fixed effects** combine to describe mean trajectory over time (can have fixed slopes up to ***n*** **– 1**):
  - Fixed Intercept = Predicted mean level (at time 0)
  - Fixed Linear Time = Mean linear rate of change (at time 0)
  - Fixed Quadratic Time = Half of mean acceleration/deceleration in linear rate of change (2*quad is how the linear time slope changes per unit time if quadratic is highest order fixed effect)

- **Polynomial random effects** (individual deviations from the fixed effect) describe individual differences in those change parameters (can have random slopes up to ***n*** **– 2**):
  - Random Intercept = BP variance in level (at time 0)
  - Random Linear Time = BP variance in linear time slope (at time 0)
  - Random Quadratic Time = BP variance in half the rate of acceleration/deceleration of linear time slope (across all time if quadratic is highest-order random effect)

Random Quadratic Time Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + \beta_{2i}Time_{ti}^2 + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

- β_{0i}: **Intercept** for person *i*
- γ_{00}: Fixed (mean) **Intercept**
- U_{0i}: Random (Deviation) **Intercept**

$\beta_{1i} = \gamma_{10} + U_{1i}$

- β_{1i}: **Linear Slope** for person *i*
- γ_{10}: Fixed (mean) **Linear Slope**
- U_{1i}: Random (Deviation) **Linear Slope**

$\beta_{2i} = \gamma_{20} + U_{2i}$

- β_{2i}: **Quad Slope** for person *i*
- γ_{20}: Fixed (mean) **Quad Slope**
- U_{2i}: Random (Deviation) **Quad Slope**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (*n*):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

Need $n = 4$ occasions to fit random quadratic time model

Example Sequence for Testing Fixed and Random Polynomial Effects of Time

<u>Build up fixed and random effects simultaneously:</u>

1. Empty Means, Random Intercept → to calculate ICC
2. Fixed Linear, Random Intercept → check fixed linear *p*-value
3. Random Linear → check −2ΔLL(df≈2) for random linear variance
4. Fixed Quadratic, Random Linear → check fixed quadratic *p*-value
5. Random Quadratic → check −2ΔLL(df ≈ 3) for random quadratic variance
6. …….

<u>*** In general: Can use **REML** for all models, so long as you:</u>

→ Test significance of new **fixed** effects by their ***p*-values**

→ Test significance of new **random** effects in separate step by **−2ΔLL**

→ Also see if AIC and BIC are smaller when adding random effects

Conditionality of Polynomial Random Effects

- We saw previously that lower-order fixed effects of time are conditional on higher-order polynomial fixed effects of time
- The same is true for polynomial **random effects of time**:
  - ***Random Intercept Only?***
    - Random Intercept = BP variance *for any occasion* in predicted mean Y (= variance in grand mean because individual lines are parallel)
  - ***Add Random Linear Time?***
    - Random Intercept = **now** BP variance *at time=0* in predicted mean Y *(would be different if time was centered elsewhere)*
    - Random Linear Time = BP variance *across all occasions* in linear rate of change *(would be the same if time was centered elsewhere)*
  - ***Add Random Quadratic Time?***
    - Random Intercept = still BP variance *at time=0* in predicted mean Y
    - Random Linear Time = **now** BP variance *at time=0* in linear rate of change *(would be different if time was centered elsewhere)*
    - Random Quadratic Time = BP variance *across all occasions* in half of accel/decel of change *(would be the same if time was centered elsewhere)*

Random Effects Allowed by #Occasions

	Data		G Matrix	R Matrix	Variance Model # Parameters
n=2 occasions **3** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 \end{bmatrix}$ **Random Intercept only**	$\begin{bmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}$	**2**
n=3 occasions **6** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & & \\ \sigma_{21} & \sigma_2^2 & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 & \\ \tau_{U_{01}} & \tau_{U_1}^2 \end{bmatrix}$ **Up to 1 Random slope**	$\begin{bmatrix} \sigma_e^2 & 0 & 0 \\ 0 & \sigma_e^2 & 0 \\ 0 & 0 & \sigma_e^2 \end{bmatrix}$	**4**
n=4 occasions **10** unique pieces of information	$\begin{bmatrix} \sigma_1^2 & & & \\ \sigma_{21} & \sigma_2^2 & & \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_4^2 \end{bmatrix}$	→	$\begin{bmatrix} \tau_{U_0}^2 & & \\ \tau_{U_{01}} & \tau_{U_1}^2 & \\ \tau_{U_{02}} & \tau_{U_{12}} & \tau_{U_2}^2 \end{bmatrix}$ **Up to 2 Random slopes**	$\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$	**7**

Predicted **V** Matrix from Polynomial Random Effects Models

- **<u>Random linear model?</u>** Variance has a **quadratic** dependence on time
  - Variance will be at a minimum when time = $-\text{Cov}(U_0,U_1)/\text{Var}(U_1)$, and will increase parabolically and symmetrically over time
  - **Predicted variance** at each occasion and covariance between A and B:

$$\text{Var}(y_{time}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0,U_1)(\mathbf{time_t}) + \text{Var}(U_1)(\mathbf{time_t^2})$$

$$\text{Cov}(y_A,y_B) = \text{Var}(U_0) + \text{Cov}(U_0,U_1)(A + B) + \text{Var}(U_1)(AB)$$

- **<u>Random quadratic model?</u>** Variance has a **quartic** dependence on time

$$\text{Var}(y_{time}) = \text{Var}(e_t) + \text{Var}(U_0) + 2\text{Cov}(U_0,U_1)(\mathbf{time_t}) + \text{Var}(U_1)(\mathbf{time_t^2}) + 2\text{Cov}(U_0,U_2)(\mathbf{time_t^2}) + 2\text{Cov}(U_1,U_2)(\mathbf{time_t^3}) + \text{Var}(U_2)(\mathbf{time_t^4})$$

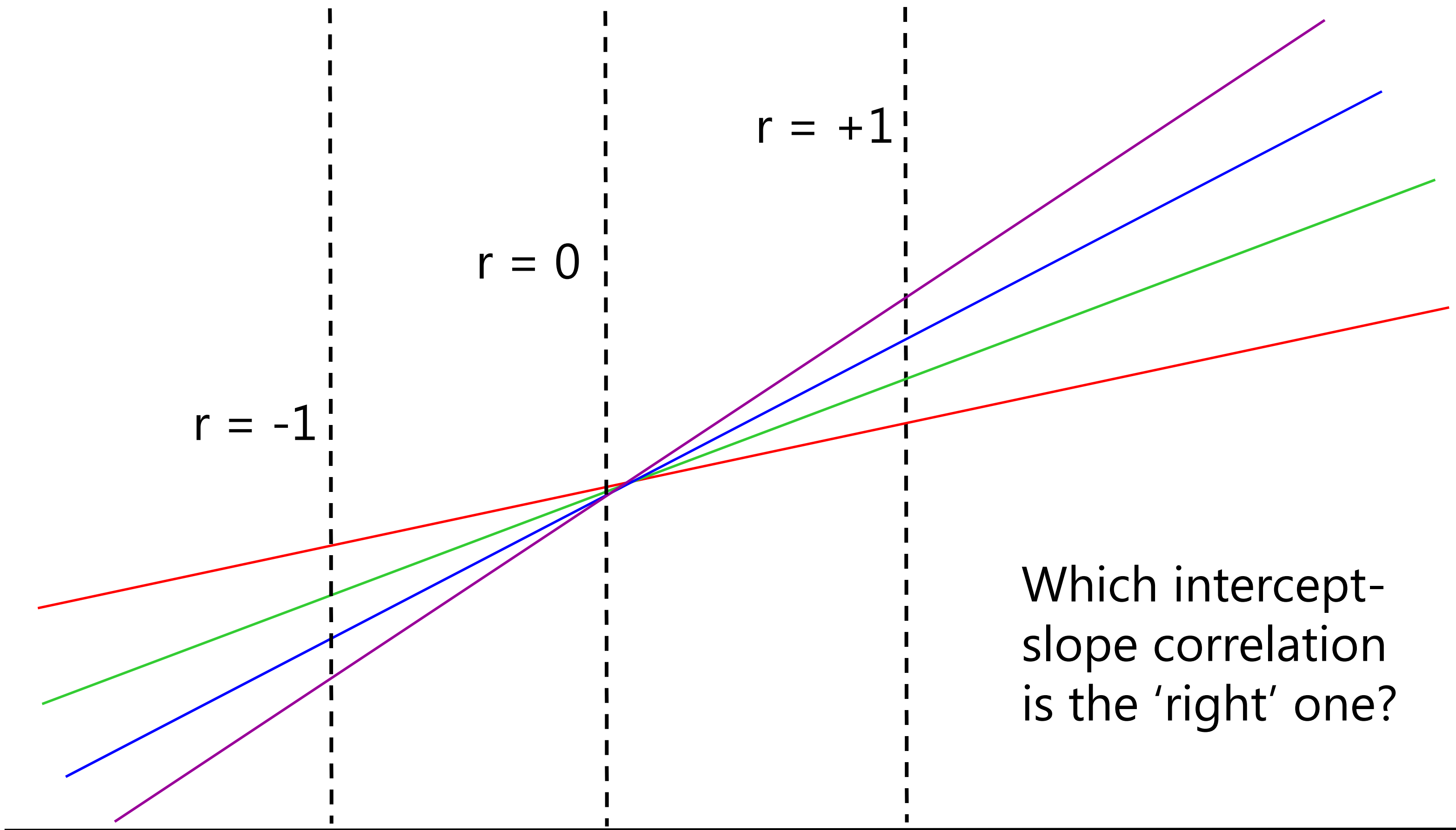
$$\text{Cov}(y_A,y_B) = \text{Var}(U_0) + \text{Cov}(U_0,U_1)(A + B) + \text{Var}(U_1)(AB) + \text{Cov}(U_0,U_2)(A^2 + B^2) + \text{Cov}(U_1,U_2)[(AB^2)+(A^2B)] + \text{Var}(U_2)(A^2B^2)$$

- *The point of the story: random effects of time are a way of allowing the variances and covariances to differ over time in specific, time-dependent patterns (that result from differential individual change over time).*

Rules for Polynomial Models (and in general for fixed and random effects)

- On the same side of the model (means or variances side), lower-order effects stay in EVEN IF NONSIGNIFICANT (for correct interpretation)
  - e.g., Significant *fixed* quadratic? Keep the *fixed* linear
  - e.g., Significant *random* quadratic? Keep the *random* linear
- Also remember—you can have a significant random effect EVEN IF the corresponding fixed effect is not significant (keep it anyway):
  - e.g., Fixed linear not significant, but random linear is significant?
    → No linear change *on average*, but significant individual differences in change
- Language: A random effect supersedes a fixed effect:
  - If Fixed = intercept, linear, quad; Random = intercept, linear, quad?
    - Call it a "Random quadratic model" (implies everything beneath those terms)
  - If Fixed = intercept, linear, quad; Random = intercept, linear?
    - Call it a "Fixed quadratic, random linear model" (distinguishes no random quad)
- Intercept-slope correlation depends largely on centering of time…

Correlation between Random Intercept and Random Linear Slope depends on time 0

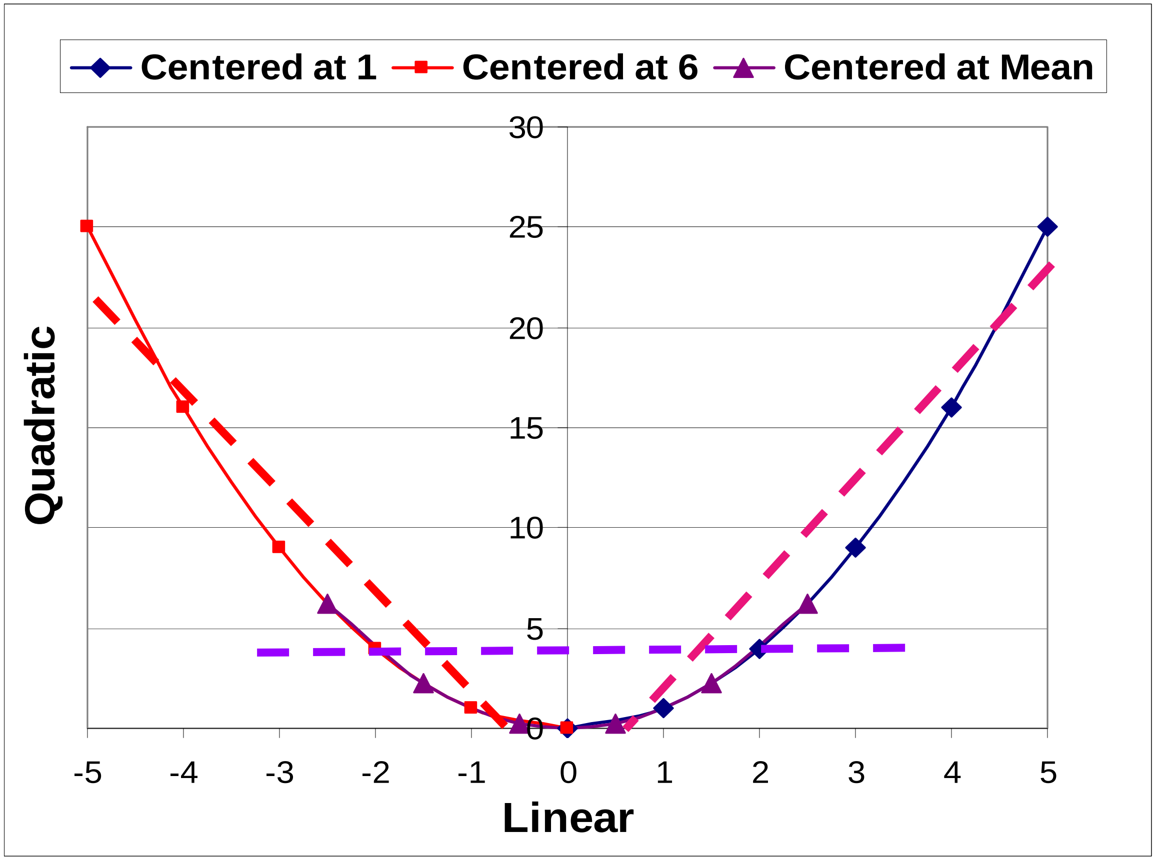


Which intercept-slope correlation is the 'right' one?

!! Nonparallel lines will eventually cross.

Correlations among polynomial slopes

Session Centered at 1:			Session Centered at 6:			Session Centered at Mean:		
Session	Linear	Quadratic	Session	Linear	Quadratic	Session	Linear	Quadratic
1	0	0	1	-5	25	1	-2.5	6.25
2	1	1	2	-4	16	2	-1.5	2.25
3	2	4	3	-3	9	3	-0.5	0.25
4	3	9	4	-2	4	4	0.5	0.25
5	4	16	5	-1	1	5	1.5	2.25
6	5	25	6	0	0	6	2.5	6.25



Correlations among polynomial effects of time can be induced by centering time near the start or near the end.

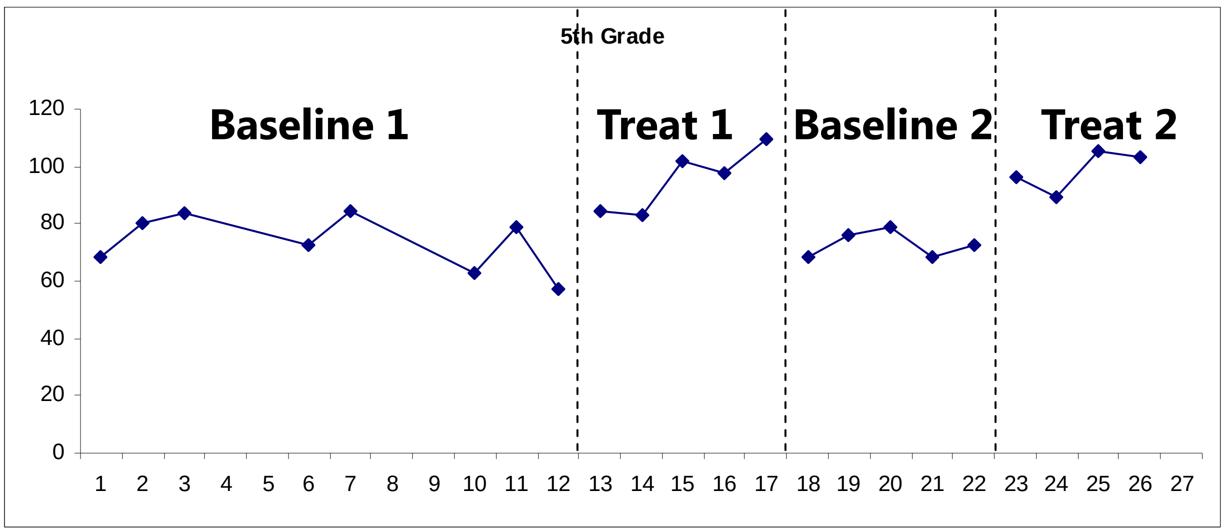
Therefore, these correlations will be *most* interpretable when centering time at its mean instead.

Summarizing so far…

- Modeling within-person change involves specifying effects of time for both sides of the model
  - **Fixed effects in model for the means:**
    - What kind of change am I observing on average?
    - What kind of trajectory will reproduce those means?
  - **Random effects (and residuals) in model for the variances:**
    - What kind of individual differences in change am I observing?
    - How many random effects do I need to reproduce the observed pattern of variances and covariances over time?
- One option: Polynomial models (linear, quadratic, cubic)
  - Terms work together to describe non-linear trajectories
  - Careful with the covariances among random effects, though
- Coming next: Piecewise slopes and truly nonlinear change…

Other Random Effects Models of Change

- **Piecewise models**: Discrete slopes for discrete phases of time
  - Separate terms describe sections of overall trajectories
  - Useful for examining change in intercepts and slopes before/after discrete events (changes in policy, interventions)
  - **Must know where the break point is ahead of time!**



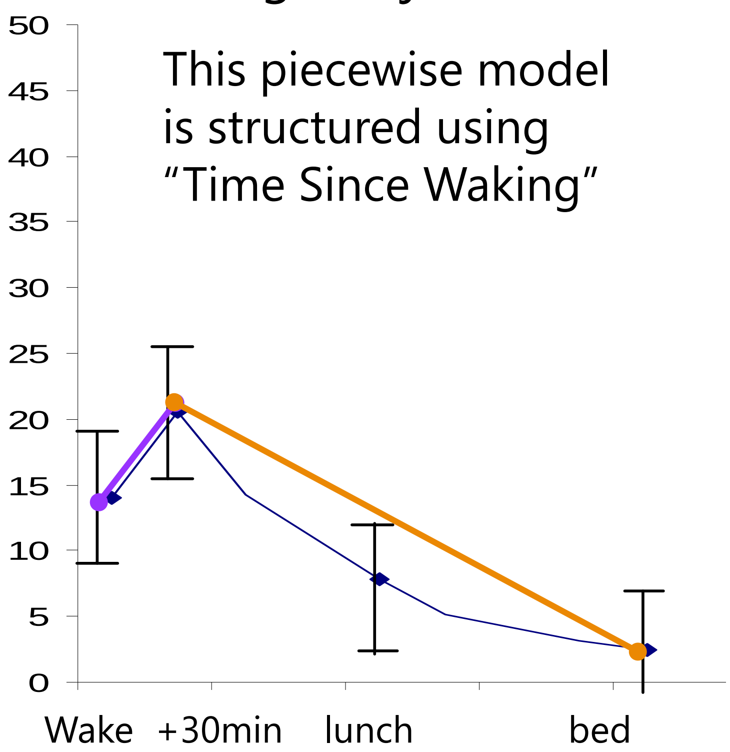
Piecewise Model:

4 slopes
(one per phase)

3 "jumps"
(shift in intercept between phases)

Example of Daily Cortisol Fluctuation: Morning Rise and Afternoon Decline

Average Trajectories



SAS Code to create two piecewise slopes from continuous time of day in stacked data:

```
IF occasion=1 THEN DO;
    P1=0;              P2=0; END;
IF occasion=2 THEN DO;
    P1= time2-time1; P2=0; END;
IF occasion=3 THEN DO;
    P1= time2-time1; P2=time3-time2; END;
IF occasion=4 THEN DO;
    P1= time2-time1; P2=time4-time2; END;
```

Note that a quadratic slope may be necessary for the afternoon decline slope!

Random Two-Slope Piecewise Model

Level 1: $\mathbf{y_{ti} = \beta_{0i} + \beta_{1i}Slope1_{ti} + \beta_{2i}Slope2_{ti} + e_{ti}}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

- β_{0i}: **Intercept** for person *i*
- γ_{00}: Fixed (mean) **Intercept**
- U_{0i}: Random (Deviation) **Intercept**

$\beta_{1i} = \gamma_{10} + U_{1i}$

- β_{1i}: **Slope1** for person *i*
- γ_{10}: Fixed (mean) **Slope1**
- U_{1i}: Random (Deviation) **Slope1**

$\beta_{2i} = \gamma_{20} + U_{2i}$

- β_{2i}: **Slope2** for person *i*
- γ_{20}: Fixed (mean) **Slope2**
- U_{2i}: Random (Deviation) **Slope2**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

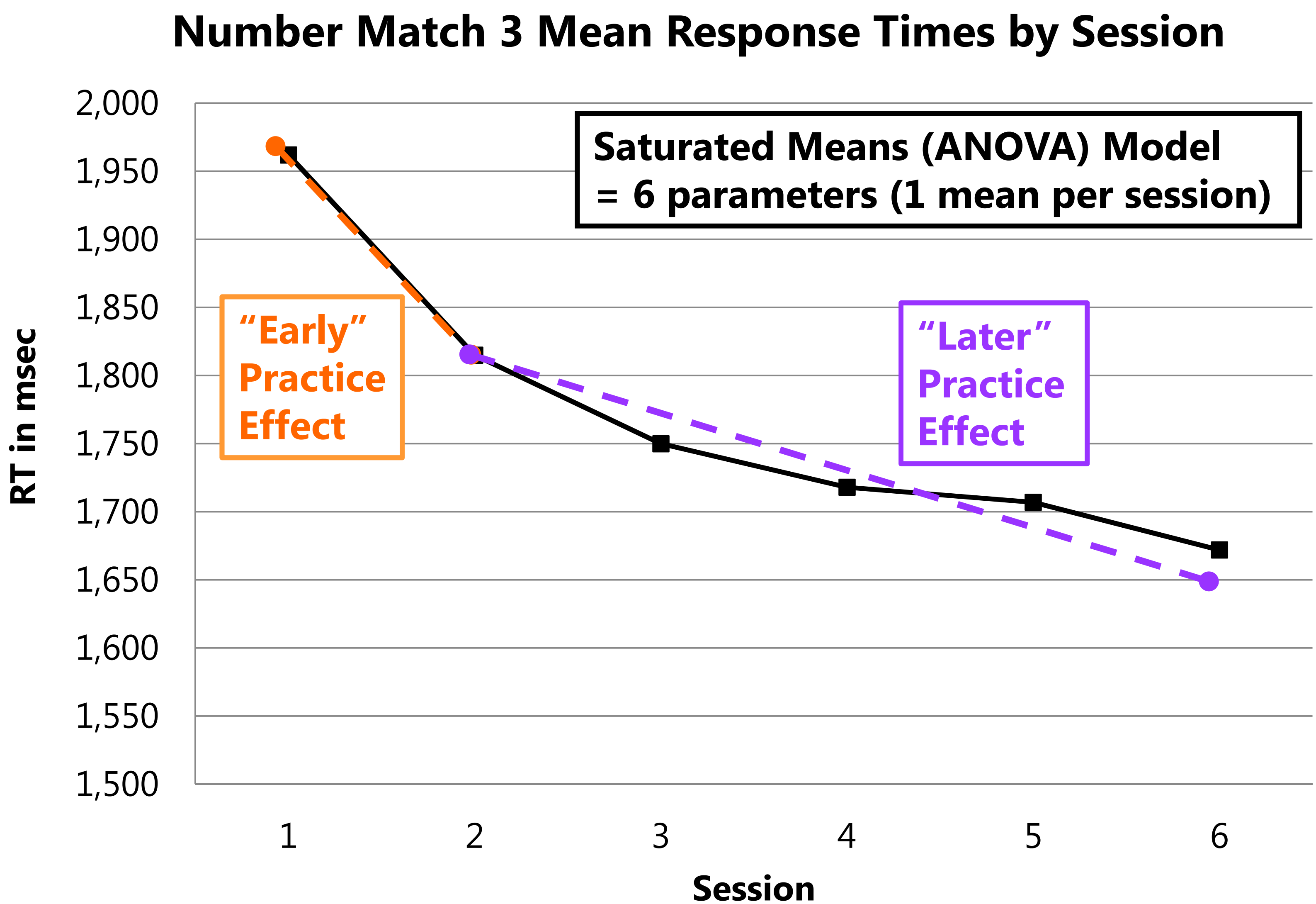
Number of Possible Slopes by Number of Occasions (n):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

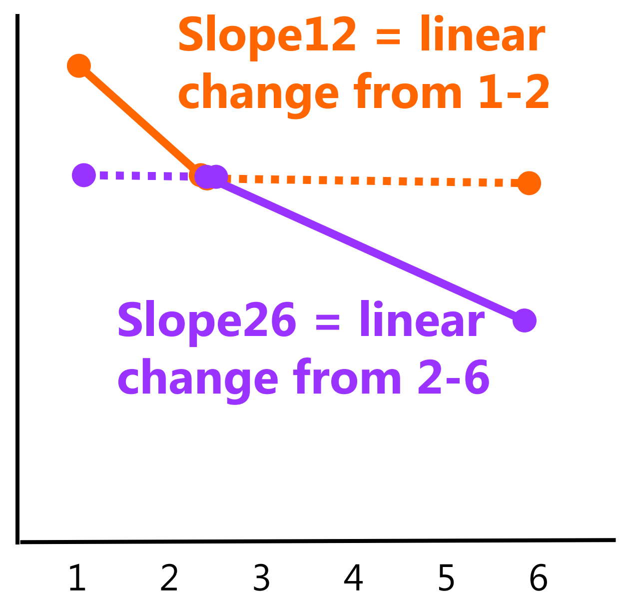
Need n = 4 occasions to fit random two-slope model

What kind of piecewise model could predict our example data mean change across sessions?



Piecewise Models: Two Direct Slopes

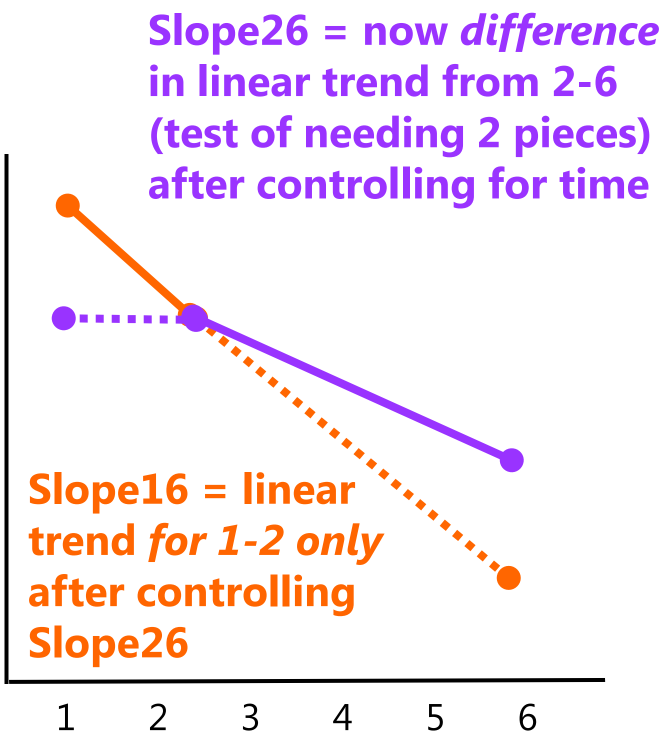
- "Early Practice Slope" and "Later Practice Slope"
- Use to specify slopes through each discrete phase directly
- Session (1-6) gets recoded into 2 new time predictor variables, as shown below:



Session	1	2	3	4	5	6
Early Practice → Slope12 =	0	1	1	1	1	1
Later Practice → Slope26 =	0	0	1	2	3	4

Piecewise Models: Slope +Deviation Slope

- "Linear Time Slope" and "Deviation Slope"
- Use to test if multiple slopes are needed
- Initial slope predictor is coded differently, second slope predictor is same:



Session		1	2	3	4	5	6
Time	→ Slope16 =	0	1	2	3	4	5
Deviation	→ Slope26 =	0	0	1	2	3	4

2 Direct Slopes Model: Random Effects

- Parameters directly **represent each part** of trajectory:

- **Fixed effects** for mean change over time (3):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope12 = expected linear rate of change from 1 to 2
  - Fixed Slope26 = expected linear rate of change from 2 to 6

- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level when both slopes = 0 (at Session 1)
  - Random Slope12 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in linear slope from 2 to 6

Slope + Deviation Slope: Random Effects

- Parameters directly **differences across parts** of trajectory:

- **Fixed effects** for mean change over time (3):
  - Fixed Intercept = expected Y when both slopes = 0 (Session 1)
  - Fixed Slope16 = expected linear rate of change from 1 to 2 (after controlling for slope26)
  - Fixed Slope26 = expected **extra** linear rate of change from 2 to 6 (after controlling for slope16, which is just time)

- Leads to possible **random effects** (up to 3 var+3 cov):
  - Random Intercept = BP variance in expected level when both slopes = 0 (at Session 1)
  - Random Slope16 = BP variance in linear slope from 1 to 2
  - Random Slope26 = BP variance in **extra** linear slope from 2 to 6

Saturated Means via Piecewise Slopes Models

- You can fit **fixed** piecewise slopes up to n–1, but only **random** piecewise slopes up to n–2:
  - 3 occasions? up to 2 fixed pieces, but only 1 random piece
  - 4 occasions? up to 3 fixed pieces, but only 2 random pieces
  - n–1 fixed pieces will perfectly reproduce observed means

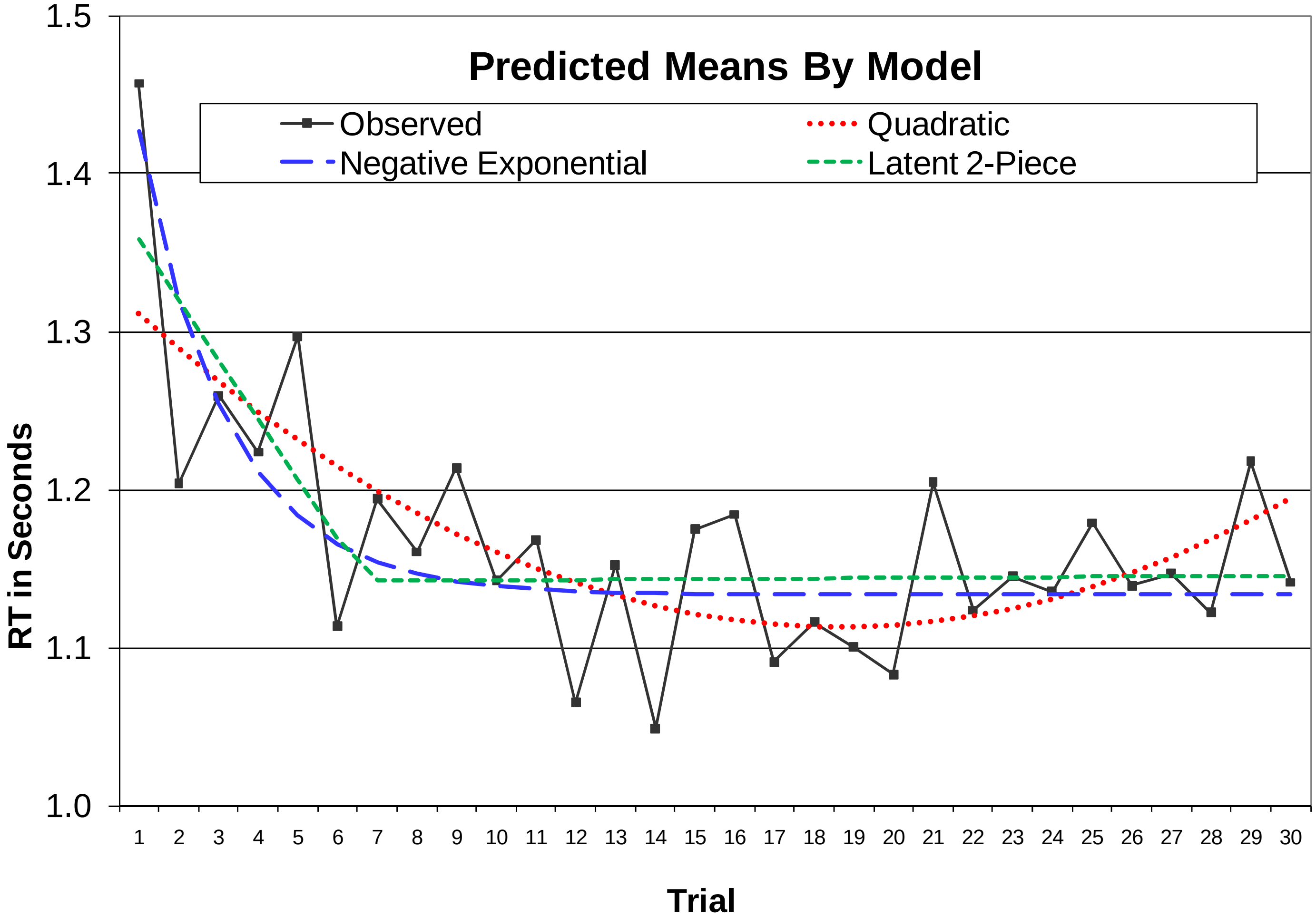
- Given this constraint (and balanced data), you should consider some of the ACS models as well:
  - Example: n=3 → Model for the means = 2 fixed pieces, Model for the Variances could be….
    - UN, CSH, CS (Random Intercept Only), Random Intercept + Random Slope12, OR Random Intercept + Random Slope23
    - Everything is nested within UN; can also use AIC and BIC to choose

Summary: Piecewise Slopes Models

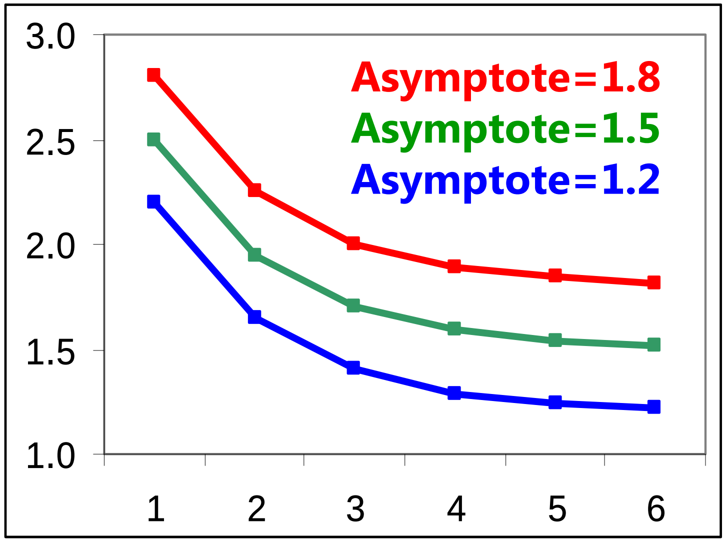
- Piecewise models are useful for discontinuous trajectories (empirically or based on the study design)
  - Use slope + deviation slope(s) to test if > 1 slope is necessary
- If all effects are random, the slope + deviation slope and the direct slopes versions of the models will be equivalent
  - Select the one that has the random effects variance you want to predict
- Keep all the pieces in the model (even if non-significant) in order to maintain a correct interpretation of each
- Each piece can be linear or non-linear as needed
  - e.g., piece1 + piece2 + piece2^2 → linear, then non-linear trajectory
- You may also need to test for a 'drop' or 'jump' in intercept at the break point in addition to change in slope, data permitting
  - Planning on doing piecewise models? They can be tricky…
    PLEASE let me help you set up the predictors to do so!

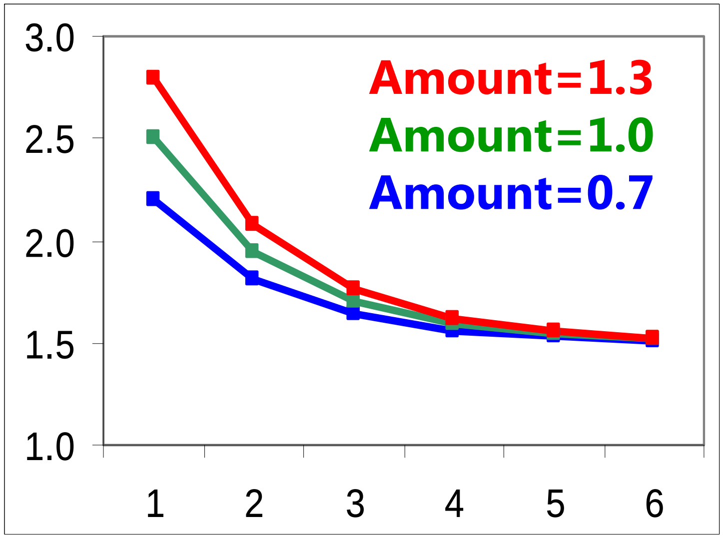
Other Random Effects for Change

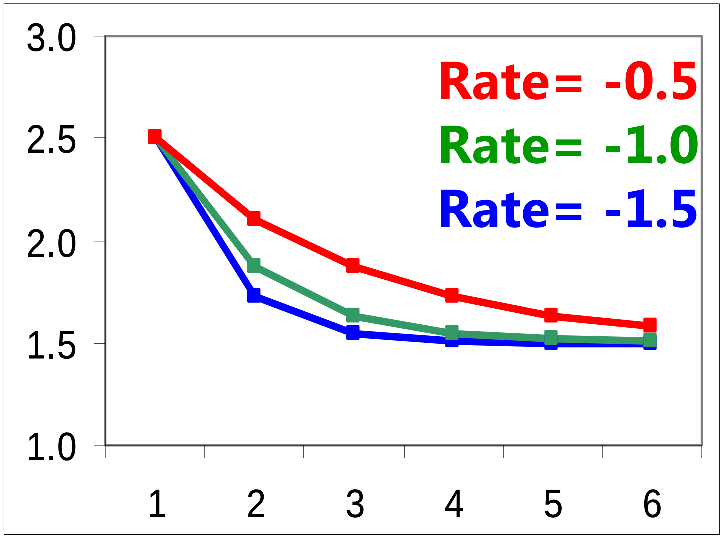
- **Truly nonlinear models**: Non-additive terms to describe change
  - Models can include **asymptotes** (so change can "shut off" as needed)
  - Include **power** and **exponential** functions (see chapter 6 for references)



(Negative) Exponential Model Parameters

1) 

2) 

3) 

1) Different **Asymptotes**, same amount and rate

2) Different **Amounts**, same asymptote and rate

3) Different **Rates**, same asymptote and amount

Exponential Models

- The name positive or negative reflects whether the data are moving away or towards asymptote
  - *Accelerating* trajectory (up or down) = “positive” exponential
  - *Decelerating* trajectory (up or down) = “negative” exponential

- Amount reflects distance from asymptote to time 0, multiplied by exp(rate*time)
  - Decrease across time to asymptote = positive amount
  - Increase across time to asymptote = negative amount

- Amount can also be replaced by an intercept
  - Asymptote + Amount = Intercept

- Cannot be estimated in SAS PROC MIXED given its nonlinear parameters (use SAS PROC NLMIXED instead)

Exponential Model (3 Random Effects)

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}*\exp(\beta_{2i}*Time_{ti}) + e_{ti}$

Level 2 Equations (one per β):

$\beta_{0i} = \gamma_{00} + U_{0i}$

Asymptote for person *i* — Fixed (mean) **Asymptote** — Random (Deviation) **Asymptote**

$\beta_{1i} = \gamma_{10} + U_{1i}$

Amount for person *i* — Fixed (mean) **Amount** — Random (Deviation) **Amount**

$\beta_{2i} = \gamma_{20} + U_{2i}$

Rate for person *i* — Fixed (mean) **Rate** — Random (Deviation) **Rate**

Fixed Effect Subscripts:

1st = which Level 1 term

2nd = which Level 2 term

Number of Possible Slopes by Number of Occasions (*n*):

Fixed slopes = $n - 1$

Random slopes = $n - 2$

Also need 4 occasions to fit random exponential model

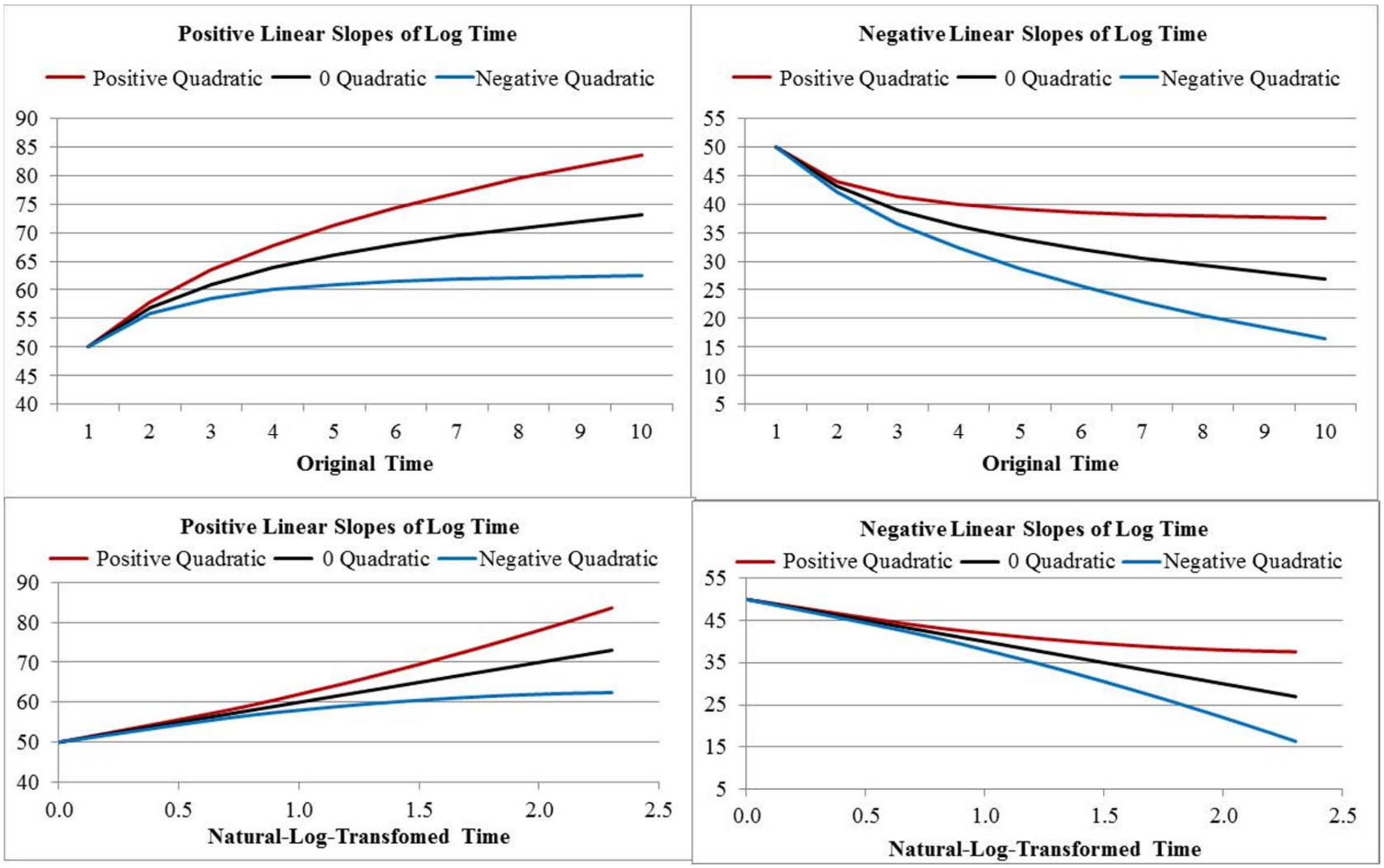
(Likely need way more occasions to find U_{2i}, though)

Nonlinear Models

- Not all forms of change fit polynomial models
  - What goes up must come back down (and vice-versa)
  - Sometimes change needs to "shut off" (need asymptotes)
- Many kinds of truly nonlinear models can be used for longitudinal data
  - Linear in variables vs. linear in parameters (exp → nonlinear)
  - Logistic, power, exponential… see end of chapter 6 for ideas
- Require extra steps to evaluate estimation quality
  - Start values are needed, especially for random variances
  - Check that "gradient" values are as close to 0 as possible (partial first derivative of that parameter in LL function)

How to Mimic an Exponential Model

If you need to use REML, a predictor of natural-log-transformed time may be a good substitute for a truly nonlinear model



A linear effect of log time (black lines) predicts an exponential curve across *original* time.

Quadratic effects of log time (red or blue lines) can speed up or slow down the curve.

Bottom: There is a linear relationship between log-time and the outcome.

Which change family should I choose?

- Within a given family, nested models can usually be compared to judge the need for each parameter
  - e.g., linear vs. quadratic? One slope vs. two slopes?
  - Usual nested model comparison rules apply (*p*-values for fixed effects, −2ΔLL tests for random effects)
  - When using REML, you can test absolute fit of each side separately if you have balanced data to see if you are "there yet"

- Between families, however, alternative models of change may not be nested, so deciding among them can be tricky
  - e.g., quadratic vs. two-slope vs. log time vs. exponential?
  - Use ML AIC and BIC to see what is "preferred" across the families
  - In balanced data, you can also compare each alternative to a saturated means, UN model using ML as test of absolute fit
  - Also consider plausibility of alternative models in terms of both data predictions and theoretical predictions in deciding