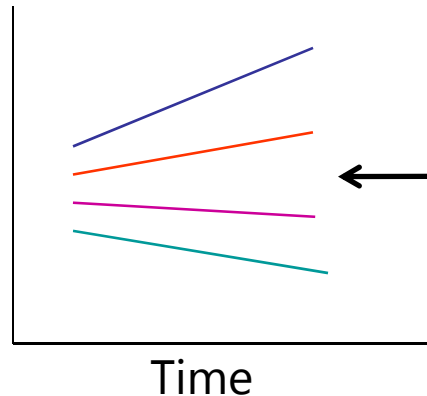


# Describing Within-Person Fluctuation over Time using Alternative Covariance Structures

- Topics:
  - **The Big Picture**
  - ACS models using the **R** matrix only
  - Introducing the **G**, **Z**, and **V** matrices
  - ACS models combining the **G** and **R** matrices

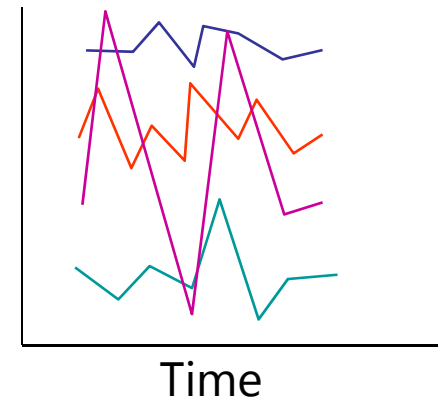
# Modeling Change vs. Fluctuation

Pure WP Change



Our focus  
right now

Pure WP Fluctuation



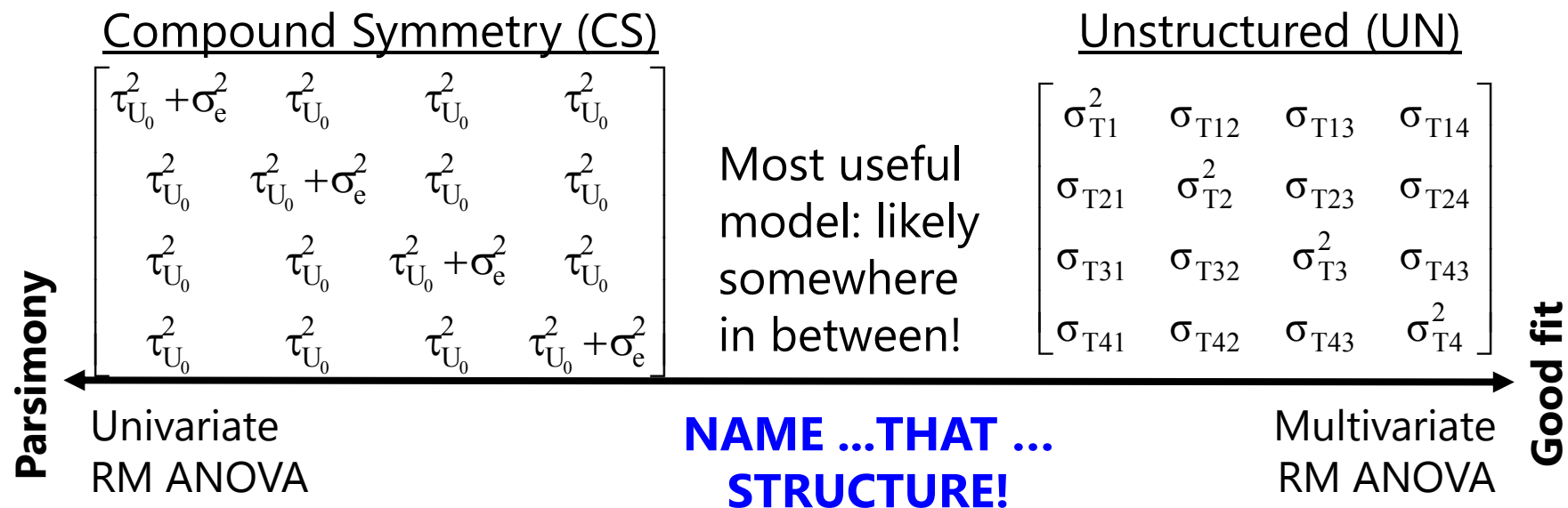
## Model for the Means:

- WP Change → describe pattern of *average* change (over "time")
- **WP Fluctuation** → \*may\* not need anything (if no systematic change)

## Model for the Variances:

- WP Change → describe *individual differences* in change (random effects)  
→ this allows variances and covariances to differ over time
- **WP Fluctuation** → describe pattern of variances and covariances over time

# Big Picture Framework: Models for the Variance in Longitudinal Data



***What is the pattern of variance and covariance over time?***

CS and UN are just two of the many, many options available within MLM, including *random effects models* (for change) and *alternative covariance structure models* (for fluctuation).

# Alternative Covariance Structure Models

- Useful in predicting patterns of variance and covariance that arise from fluctuation in the outcome over time:
  - **Variances**: Same (homogeneous) or different (heterogeneous)?
  - **Covariances**: Same or different? If different, what is the pattern?
    - Models with heterogeneous variances predict correlation instead of covariance
  - Often don't need any fixed effects for systematic effects of time in the model for the means (although this is always an empirical question)
- Limitations for most of the ACS models:
  - Require **equal-interval** occasions (they are based on idea of "time lag")
  - Require **balanced** time across persons (no intermediate time values)
  - But **do not require complete data** (unlike when CS and UN are estimated via least squares in ANOVA instead of ML/REML in MLM)
- ACS models do require some new terminology to introduce...

# Likelihood-Based Model Comparisons

- **Relative model fit** is indexed by a “**deviance**” statistic → **-2LL**
  - **-2LL indicates BADNESS of fit, so smaller values = better models**
- **Nested models are compared using their deviance values: -2ΔLL Test** (i.e., Likelihood Ratio Test, Deviance Difference Test)
  1. Calculate  $-2\Delta LL$ :  $(-2LL_{\text{fewer}}) - (-2LL_{\text{more}})$
  2. Calculate  $\Delta df$ :  $(\# \text{Parms}_{\text{more}}) - (\# \text{Parms}_{\text{fewer}})$
  3. Compare  $-2\Delta LL$  to  $\chi^2$  distribution with  $df = \Delta df$   
*CHIDIST in excel will give exact p-values for the difference test; so will STATA*

1. & 2. must be positive values!
- Nested or non-nested models can also be compared by **Information Criteria** that reflect **-2LL** AND # parameters used and/or sample size
  - **AIC** = Akaike IC = **-2LL** +  $2 * (\# \text{parameters})$
  - **BIC** = Bayesian IC = **-2LL** +  $\log(N) * (\# \text{parameters})$  → penalty for complexity
  - No significance tests or critical values, just “smaller is better”

# Two Families of ACS Models

- So far, we've referred to the variance and covariance matrix of the longitudinal outcomes as the **R** matrix
  - We now refer to these as “**R-only models**” (use **REPEATED** statement only)
  - Although the **R** matrix is actually specified per individual, ACS models usually assume the same **R** matrix for everyone
  - **R** matrix is symmetric with dimensions  $n \times n$ , in which  $n = \#$  occasions per person (although people can have missing data, the same set of *possible* occasions is required across people to use most **R-only** models)
- **3 other matrices we'll see in “G and R combined” ACS models:**
  - **G** = matrix of random effects variances and covariances (stay tuned)
  - **Z** = matrix of values for predictors that have random effects (stay tuned)
  - **V** = symmetric  $n \times n$  matrix of **total** variance and covariance over time
    - If the model includes random effects, then **G** and **Z** get combined with **R** to make **V** as  $\mathbf{V} = \mathbf{ZGZ}^T + \mathbf{R}$  (accomplished by adding the **RANDOM** statement)
    - If the model does NOT include random effects in **G**, then  $\mathbf{V} = \mathbf{R}$ ... so, **R-only**

# Review: Covariances and Correlations

$$\text{Correlation}_{y1,y2} = \frac{\text{Covariance}_{y1,y2}}{\sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}}$$

$$\text{Covariance}_{y1,y2} = \text{Correlation}_{y1,y2} * \sqrt{\text{Variance}_{y1}} * \sqrt{\text{Variance}_{y2}}$$

- Given the standard deviation (as  $\sqrt{\text{Variance}}$ ) at each occasion, either the correlation and covariance can be calculated given the other
- ACS models with **homogeneous variances** tend to be specified in terms of **variance and covariance**
  - Given same variance over time, same covariance → same correlation
- ACS models with **heterogeneous variance** tend to be specified in terms of **variance and correlation**
  - Different variances over time → different covariances over time, even if the correlation is the same (so only correlation is estimated directly)

# Describing Within-Person Fluctuation over Time using Alternative Covariance Structures

- Topics:
  - The Big Picture
  - **ACS models using the  $\mathbf{R}$  matrix only**
  - Introducing the  $\mathbf{G}$ ,  $\mathbf{Z}$ , and  $\mathbf{V}$  matrices
  - ACS models combining the  $\mathbf{G}$  and  $\mathbf{R}$  matrices



# R-Only ACS Models

- The **R-only** models to be presented next are all specified using the **REPEATED** statement only (no RANDOM statement)
- They are explained by showing their predicted **R** matrix, which provides the **total** variances and covariances across occasions
  - Total variance per occasion on diagonal
  - Total covariances across occasions on off-diagonals
  - I've included in " " the labels SAS uses for each parameter
- Correlations across occasions can be calculated given variances and covariances, which would be shown in the **RCORR** matrix (available in SAS PROC MIXED)
  - 1's on diagonal (standardized variables), correlations on off-diagonal
- **Unstructured (TYPE=UN) will always fit best by -2LL**
  - All ACS models are nested within Unstructured (UN = the data)
  - Goal: find an ACS model that is **simpler** but **not worse fitting** than UN

# R-Only ACS Models: CS/CSH

## • Compound Symmetry: TYPE=CS

➤ 2 parameters:

- **1 “residual” variance  $\sigma_e^2$**
- **1 “CS” covariance**  
across occasions

➤ Constant total variance:  $CS + \sigma_e^2$

➤ Constant total covariance: CS

$$\begin{bmatrix} CS + \sigma_e^2 & CS & CS & CS \\ CS & CS + \sigma_e^2 & CS & CS \\ CS & CS & CS + \sigma_e^2 & CS \\ CS & CS & CS & CS + \sigma_e^2 \end{bmatrix}$$

## • Compound Symmetry Heterogeneous: TYPE=CSH

➤  $n+1$  parameters:

- **$n$  separate “Var( $n$ )”  
total variances  $\sigma_{Tn}^2$**
- **1 “CSH” total correlation**  
across occasions

$$\begin{bmatrix} \sigma_{T1}^2 & CSH\sigma_{T1}\sigma_{T2} & CSH\sigma_{T1}\sigma_{T3} & CSH\sigma_{T1}\sigma_{T4} \\ CSH\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & CSH\sigma_{T2}\sigma_{T3} & CSH\sigma_{T2}\sigma_{T4} \\ CSH\sigma_{T3}\sigma_{T1} & CSH\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & CSH\sigma_{T3}\sigma_{T4} \\ CSH\sigma_{T4}\sigma_{T1} & CSH\sigma_{T4}\sigma_{T2} & CSH\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

➤ Separate total variances are estimated directly

➤ Still constant total correlation: CSH (but has non-constant covariances)

# R-Only ACS Models: AR1/ARH1

## • 1<sup>st</sup> Order Auto-Regressive: TYPE=AR(1)

➤ 2 parameters:

- **1 constant total variance**  
 $\sigma_T^2$  (misabeled “residual”)
- **1 “AR1” total auto-correlation  $r_T$**   
across occasions

$$\begin{bmatrix} \sigma_T^2 & r_T^1 \sigma_T^2 & r_T^2 \sigma_T^2 & r_T^3 \sigma_T^2 \\ r_T^1 \sigma_T^2 & \sigma_T^2 & r_T^1 \sigma_T^2 & r_T^2 \sigma_T^2 \\ r_T^2 \sigma_T^2 & r_T^1 \sigma_T^2 & \sigma_T^2 & r_T^1 \sigma_T^2 \\ r_T^3 \sigma_T^2 & r_T^2 \sigma_T^2 & r_T^1 \sigma_T^2 & \sigma_T^2 \end{bmatrix}$$

- $r_T^1$  is lag-1 correlation,  $r_T^2$  is lag-2 correlation,  $r_T^3$  is lag-3 correlation....

## • 1<sup>st</sup> Order Auto-Regressive Heterogeneous: TYPE=ARH(1)

➤  $n+1$  parameters:

- **$n$  separate “Var( $n$ )”**  
**total variances  $\sigma_{Tn}^2$**
- **1 “ARH1” total auto-**  
**correlation  $r_T$**  across occasions

$$\begin{bmatrix} \sigma_{T1}^2 & r_T^1 \sigma_{T1} \sigma_{T2} & r_T^2 \sigma_{T1} \sigma_{T3} & r_T^3 \sigma_{T1} \sigma_{T4} \\ r_T^1 \sigma_{T2} \sigma_{T1} & \sigma_{T2}^2 & r_T^1 \sigma_{T2} \sigma_{T3} & r_T^2 \sigma_{T2} \sigma_{T4} \\ r_T^2 \sigma_{T3} \sigma_{T1} & r_T^1 \sigma_{T3} \sigma_{T2} & \sigma_{T3}^2 & r_T^1 \sigma_{T3} \sigma_{T4} \\ r_T^3 \sigma_{T4} \sigma_{T1} & r_T^2 \sigma_{T4} \sigma_{T2} & r_T^1 \sigma_{T4} \sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

- $r_T^1$  is lag-1 correlation,  $r_T^2$  is lag-2 correlation,  $r_T^3$  is lag-3 correlation....

# R-Only ACS Models: TOEP<sub>n</sub>/TOEPH<sub>n</sub>

## • Toeplitz(*n*): TYPE=TOEP(*n*)

➤ *n* parameters:

- **1 constant total variance**  
 $\sigma_T^2$  (misabeled “residual”)
- ***n*–1 “TOEP(lag)”  $c_{Tn}$  banded total covariances** across occasions
- $c_{T1}$  is lag-1 covariance,  $c_{T2}$  is lag-2 covariance,  $c_{T3}$  is lag-3 covariance....

$$\begin{bmatrix} \sigma_T^2 & & & \\ c_{T1} & \sigma_T^2 & & \\ c_{T2} & c_{T1} & \sigma_T^2 & \\ c_{T3} & c_{T2} & c_{T1} & \sigma_T^2 \end{bmatrix}$$

## • Toeplitz Heterogeneous(*n*): TYPE=TOEPH(*n*)

➤ *n* + (*n*–1) parameters:

- ***n* separate “Var(*n*)” total variances  $\sigma_{Tn}^2$**
- ***n*–1 “TOEPH(lag)”  $r_{Tn}$  banded total correlations** across occasions

- $r_{T1}$  is lag-1 correlation,  $r_{T2}$  is lag-2 correlation,  $r_{T3}$  is lag-3 correlation....

$$\begin{bmatrix} \sigma_{T1}^2 & r_{T1}\sigma_{T1}\sigma_{T2} & r_{T2}\sigma_{T1}\sigma_{T3} & r_{T3}\sigma_{T1}\sigma_{T4} \\ r_{T1}\sigma_{T2}\sigma_{T1} & \sigma_{T2}^2 & r_{T1}\sigma_{T2}\sigma_{T3} & r_{T2}\sigma_{T2}\sigma_{T4} \\ r_{T2}\sigma_{T3}\sigma_{T1} & r_{T1}\sigma_{T3}\sigma_{T2} & \sigma_{T3}^2 & r_{T1}\sigma_{T3}\sigma_{T4} \\ r_{T3}\sigma_{T4}\sigma_{T1} & r_{T2}\sigma_{T4}\sigma_{T2} & r_{T1}\sigma_{T4}\sigma_{T3} & \sigma_{T4}^2 \end{bmatrix}$$

# Comparing **R**-only ACS Models

- Baseline models: **CS = simplest, UN = most complex**
  - Relative to CS, more complex models fit “better” or “not better”
  - Relative to UN, less complex models fit “worse” or “not worse”
- Other rules of nesting and model comparisons:
  - Homogeneous variance models are nested within heterogeneous variance models (e.g., CS in CSH, AR1 in ARH1, TOEP in TOEPH)
  - CS and AR1 are each nested within TOEP (i.e., TOEP can become CS or AR1 through restrictions of its covariance patterns)
  - CS and AR1 are not nested (because both have 2 parameters)
  - **R**-only models differ in unbounded parameters, so can be compared using regular  $-2\Delta LL$  tests (instead of mixture  $-2\Delta LL$  tests)
  - Good idea to start by assuming heterogeneous variances until you settle on the covariance pattern, then test if het. var. are still necessary
  - When in doubt, just compare AIC and BIC (useful even with  $-2\Delta LL$  tests)

# Describing Within-Person Fluctuation over Time using Alternative Covariance Structures

- Topics:
  - The Big Picture
  - ACS models using the R matrix only
  - **Introducing the G, Z, and V matrices**
  - **ACS models combining the G and R matrices**

# The Other Family of ACS Models

- **R**-only models *directly* predict the **total** variance and covariance
- **G** and **R** models *indirectly* predict the total variance and covariance through **between-person (BP)** and **within-person (WP)** sources of variance and covariance → So, for this model:  $\mathbf{y}_{ti} = \beta_0 + \mathbf{U}_{0i} + \mathbf{e}_{ti}$ 
  - **BP** = **G** matrix of **level-2 random effect ( $\mathbf{U}_{0i}$ )** variances and covariances
    - Which effects get to be random (whose variance and covariances are then included in **G**) is specified using the **RANDOM** statement (always TYPE=UN)
    - Our ACS models have a random intercept only, so **G** is 1x1 scalar of  $[\tau_{U_0}^2]$
  - **WP** = **R** matrix of **level-1 ( $\mathbf{e}_{ti}$ ) residual** variances and covariances
    - The  $n \times n$  **R** matrix of **residual** variances and covariances **that remain** after controlling for random intercept variance is then modeled with **REPEATED**
  - **Total** = **V** =  $n \times n$  matrix of **total** variance and covariance over time that results from putting **G** and **R** together:  $\mathbf{V} = \mathbf{ZGZ}^T + \mathbf{R}$ 
    - **Z** is a matrix that holds the values of predictors with random effects, but **Z** will be an  $n \times 1$  column of 1's for now (random intercept only)

# A “Random Intercept” (**G** and **R**) Model

Total Predicted Data Matrix is called **V Matrix**

$$\begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

## Level 2, BP Variance

Unstructured **G Matrix**

(RANDOM statement)

Each person has same **1 x 1 G** matrix (no covariance across persons in two-level model)

Random Intercept  
Variance only  $\begin{bmatrix} \tau_{U_0}^2 \end{bmatrix}$

## Level 1, WP Variance

Diagonal (VC) **R Matrix**

(REPEATED statement)

Each person has same **n x n R** matrix → **equal variances and 0 covariances** across time (no covariance across persons)

Residual Variance only  $\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$



# CS as a “Random Intercept” Model

**RI and DIAG:** Total predicted data matrix is called **V matrix**, created from the **G [TYPE=UN]** and **R [TYPE=VC]** matrices as follows:

$$V = Z * G * Z^T + R = V$$

$$V = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**Z** represents  $n$  per person

**Does the end result V look familiar? It should: CS =  $\tau_{U_0}^2$**

$$\begin{bmatrix} CS + \sigma_e^2 & CS & CS & CS \\ CS & CS + \sigma_e^2 & CS & CS \\ CS & CS & CS + \sigma_e^2 & CS \\ CS & CS & CS & CS + \sigma_e^2 \end{bmatrix}$$

So if the **R-only CS model** (the simplest baseline) can be specified equivalently using **G and R**, can we do the same for the **R-only UN model** (the most complex baseline)?

Absolutely! ...*with one small catch*

# UN via a “Random Intercept” Model

**RI and UN $n-1$ :** Total predicted data matrix is called **V matrix**, created from the **G [TYPE=UN]** and **R [TYPE=UN( $n-1$ )] matrices as follows:**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & \sigma_{e12} & \sigma_{e13} & \textcircled{0} \\ \sigma_{e21} & \sigma_{e2}^2 & \sigma_{e23} & \sigma_{e24} \\ \sigma_{e31} & \sigma_{e32} & \sigma_{e3}^2 & \sigma_{e34} \\ \textcircled{0} & \sigma_{e42} & \sigma_{e43} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + \sigma_{e12} & \tau_{U_0}^2 + \sigma_{e13} & \textcircled{\tau_{U_0}^2} \\ \tau_{U_0}^2 + \sigma_{e21} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + \sigma_{e23} & \tau_{U_0}^2 + \sigma_{e24} \\ \tau_{U_0}^2 + \sigma_{e31} & \tau_{U_0}^2 + \sigma_{e32} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + \sigma_{e34} \\ \textcircled{\tau_{U_0}^2} & \tau_{U_0}^2 + \sigma_{e42} & \tau_{U_0}^2 + \sigma_{e43} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

This **RI and UN $n-1$  model** is equivalent to (makes same predictions as) the **R-only UN model**. But it shows the *residual* (not total) covariances.

Because we can't estimate all possible variances and covariances in the **R** matrix and also estimate the random intercept variance  $\tau_{U_0}^2$  in the **G** matrix, we have to eliminate the last **R** matrix covariance by setting it to 0.

Accordingly, in the **RI and UN $n-1$  model**, the random intercept variance  $\tau_{U_0}^2$  takes on the value of the covariance for the first and last occasions.

# Rationale for **G** and **R** ACS models

- Modeling WP fluctuation traditionally involves using **R** only (no **G**)  
→ **Total** BP + WP variance described by just **R** matrix (so **R=V**)
  - Correlations would still be expected even at distant time lags because of constant individual differences (i.e., the BP random intercept)
  - Resulting **R**-only model may require lots of estimated parameters as a result e.g., 8 time points? Pry need a 7-lag Toeplitz(8) model
- **Why not take out the primary reason for the covariance across occasions (the random intercept variance) and see what's left?**
  - Random intercept variance  $\tau_{U_0}^2$  in **G** → control for person mean differences
  - THEN predict just the **residual** variance/covariance in **R**, not the **total**
  - Resulting model may be more parsimonious (e.g., maybe only lag1 or lag2 occasions are still related after removing  $\tau_{U_0}^2$  as a source of covariance)
  - Has the advantage of still distinguishing BP from WP variance (useful for descriptive purposes and for calculating effect sizes later)

# Random Intercept + Diagonal R Models

**RI and DIAG:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=VC]:

*homogeneous* residual variances; **no** residual covariances

**Same fit as  
R-only CS**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**RI and DIAGH:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=UN(1)]:

*heterogeneous* residual variances; **no** residual covariances

**NOT same fit  
as R-only CSH**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & 0 & 0 & 0 \\ 0 & \sigma_{e2}^2 & 0 & 0 \\ 0 & 0 & \sigma_{e3}^2 & 0 \\ 0 & 0 & 0 & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + AR1 R Models

**RI and AR1:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=AR(1)]:

*homogeneous* residual variances; *auto-regressive lagged* residual covariances

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & r_e^1 \sigma_e^2 & r_e^2 \sigma_e^2 & r_e^3 \sigma_e^2 \\ r_e^1 \sigma_e^2 & \sigma_e^2 & r_e^1 \sigma_e^2 & r_e^2 \sigma_e^2 \\ r_e^2 \sigma_e^2 & r_e^1 \sigma_e^2 & \sigma_e^2 & r_e^1 \sigma_e^2 \\ r_e^3 \sigma_e^2 & r_e^2 \sigma_e^2 & r_e^1 \sigma_e^2 & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^3 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 \\ \tau_{U_0}^2 + r_e^3 \sigma_e^2 & \tau_{U_0}^2 + r_e^2 \sigma_e^2 & \tau_{U_0}^2 + r_e^1 \sigma_e^2 & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

**RI and ARH1:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=ARH(1)]:

*heterogeneous* residual variances; *auto-regressive lagged* residual covariances

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_e^1 \sigma_{e1} \sigma_{e2} & r_e^2 \sigma_{e1} \sigma_{e3} & r_e^3 \sigma_{e1} \sigma_{e4} \\ r_e^1 \sigma_{e2} \sigma_{e1} & \sigma_{e2}^2 & r_e^1 \sigma_{e2} \sigma_{e3} & r_e^2 \sigma_{e2} \sigma_{e4} \\ r_e^2 \sigma_{e3} \sigma_{e1} & r_e^1 \sigma_{e3} \sigma_{e2} & \sigma_{e3}^2 & r_e^1 \sigma_{e3} \sigma_{e4} \\ r_e^3 \sigma_{e4} \sigma_{e1} & r_e^2 \sigma_{e4} \sigma_{e2} & r_e^1 \sigma_{e4} \sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e1} \sigma_{e2} & \tau_{U_0}^2 + r_e^2 \sigma_{e1} \sigma_{e3} & \tau_{U_0}^2 + r_e^3 \sigma_{e1} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^1 \sigma_{e2} \sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e2} \sigma_{e3} & \tau_{U_0}^2 + r_e^2 \sigma_{e2} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^2 \sigma_{e3} \sigma_{e1} & \tau_{U_0}^2 + r_e^1 \sigma_{e3} \sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_e^1 \sigma_{e3} \sigma_{e4} \\ \tau_{U_0}^2 + r_e^3 \sigma_{e4} \sigma_{e1} & \tau_{U_0}^2 + r_e^2 \sigma_{e4} \sigma_{e2} & \tau_{U_0}^2 + r_e^1 \sigma_{e4} \sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + TOEP<sub>n-1</sub> R Models

**RI and TOEP<sub>n-1</sub>:** **V** is created from **G [TYPE=UN]** and **R [TYPE=TOEP(n-1)]**:

**homogeneous** residual variances; **banded** residual covariances

**Same fit as  
R-only TOEP(n)**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & c_{e1} & c_{e2} & 0 \\ c_{e1} & \sigma_e^2 & c_{e1} & c_{e2} \\ c_{e2} & c_{e1} & \sigma_e^2 & c_{e1} \\ 0 & c_{e2} & c_{e1} & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 \\ \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + c_{e2} \\ \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} \\ \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e2} & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

Because of  $\tau_{U_0}^2$ ,  
highest lag  
covariance in **R**  
must be set to  
0 for model to  
be identified

**RI and TOEPH<sub>n-1</sub>:** **V** is created from **G [TYPE=UN]** and **R [TYPE=TOEPH(n-1)]**:

**homogeneous** residual variances; **banded** residual covariances

**NOT same fit as  
R-only TOEPH(n)**

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_{e1}\sigma_{e1}\sigma_{e2} & r_{e2}\sigma_{e1}\sigma_{e3} & 0 \\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma_{e2}^2 & r_{e1}\sigma_{e2}\sigma_{e3} & r_{e2}\sigma_{e2}\sigma_{e4} \\ r_{e2}\sigma_{e3}\sigma_{e1} & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma_{e3}^2 & r_{e1}\sigma_{e3}\sigma_{e4} \\ 0 & r_{e2}\sigma_{e4}\sigma_{e2} & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e1}\sigma_{e2} & \tau_{U_0}^2 + r_{e2}\sigma_{e1}\sigma_{e3} & \tau_{U_0}^2 \\ \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e3} & \tau_{U_0}^2 + r_{e2}\sigma_{e2}\sigma_{e4} \\ \tau_{U_0}^2 + r_{e2}\sigma_{e3}\sigma_{e1} & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e4} \\ \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e2}\sigma_{e4}\sigma_{e2} & \tau_{U_0}^2 + r_{e1}\sigma_{e4}\sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Random Intercept + TOEP2 R Models

**RI and TOEP2:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=TOEP(2)]:

*homogeneous* residual variances; *banded* residual covariance at **lag1** only

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_e^2 & c_{e1} & 0 & 0 \\ c_{e1} & \sigma_e^2 & c_{e1} & 0 \\ 0 & c_{e1} & \sigma_e^2 & c_{e1} \\ 0 & 0 & c_{e1} & \sigma_e^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 & \tau_{U_0}^2 + c_{e1} \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + c_{e1} & \tau_{U_0}^2 + \sigma_e^2 \end{bmatrix}$$

Now we can  
test the need  
for residual  
covariances at  
higher lags

**RI and TOEPH1:**  $\mathbf{V}$  is created from  $\mathbf{G}$  [TYPE=UN] and  $\mathbf{R}$  [TYPE=TOEPH(2)]:

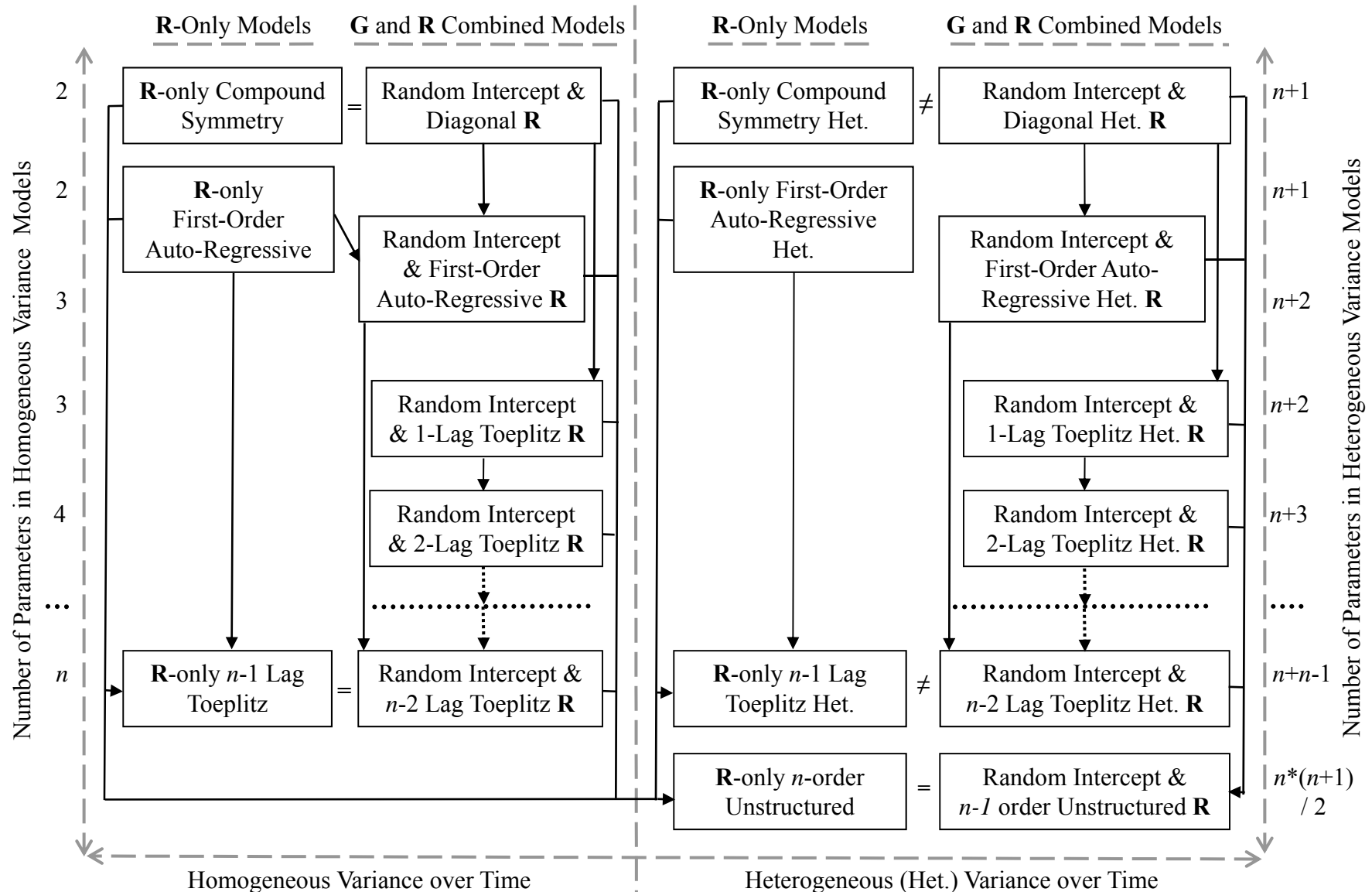
*homogeneous* residual variances; *banded* residual covariance at **lag1** only

$$\mathbf{V} = \mathbf{Z} * \mathbf{G} * \mathbf{Z}^T + \mathbf{R} = \mathbf{V}$$

$$\mathbf{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \tau_{U_0}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{e1}^2 & r_{e1}\sigma_{e1}\sigma_{e2} & 0 & 0 \\ r_{e1}\sigma_{e2}\sigma_{e1} & \sigma_{e2}^2 & r_{e1}\sigma_{e2}\sigma_{e3} & 0 \\ 0 & r_{e1}\sigma_{e3}\sigma_{e2} & \sigma_{e3}^2 & r_{e1}\sigma_{e3}\sigma_{e4} \\ 0 & 0 & r_{e1}\sigma_{e4}\sigma_{e3} & \sigma_{e4}^2 \end{bmatrix} = \begin{bmatrix} \tau_{U_0}^2 + \sigma_{e1}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e1}\sigma_{e2} & \tau_{U_0}^2 & \tau_{U_0}^2 \\ \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e1} & \tau_{U_0}^2 + \sigma_{e2}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e2}\sigma_{e3} & \tau_{U_0}^2 \\ \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e2} & \tau_{U_0}^2 + \sigma_{e3}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e3}\sigma_{e4} \\ \tau_{U_0}^2 & \tau_{U_0}^2 & \tau_{U_0}^2 + r_{e1}\sigma_{e4}\sigma_{e3} & \tau_{U_0}^2 + \sigma_{e4}^2 \end{bmatrix}$$

# Map of **R**-only and **G** and **R** ACS Models

*Arrows indicate nesting (end is more complex model)*





# Stuff to Watch Out For...

- **If using a random intercept, don't forget to drop 1 parameter in:**
  - **$n-1$  order UN  $\mathbf{R}$ :** Can't get all possible elements in  $\mathbf{R}$ , plus  $\tau_{U_0}^2$  in  $\mathbf{G}$
  - **TOEP $n-1$ :** Have to eliminate last lag covariance
- If using a random intercept...
  - Can't do RI + CS  $\mathbf{R}$ : Can't get a constant in  $\mathbf{R}$ , and then another constant in  $\mathbf{G}$
  - Can often test if random intercept helps (e.g., AR1 is nested within RI + AR1)
- If "**time**" is treated as **continuous** in the fixed effects, you will need another variable for **time** that is **categorical** to use in the syntax:
  - "Continuous Time" → on MODEL statement
  - "Categorical Time" → on CLASS and REPEATED statements
- Most alternative covariance structure models assume **time is balanced across persons with equal intervals across occasions**
  - If not, holding correlations of same lag equal doesn't make sense
  - Other structures can be used for unbalanced time
    - SP(POW)(time) = AR1 for unbalanced time (see SAS REPEATED statement for others)

# Summary: Two Families of ACS Models

- **R**-only models:
  - Specify **R** model on REPEATED statement without any random effects variances in **G** (so no RANDOM statement is used)
  - Include UN, CS, CSH, AR1, AR1H, TOEP $n$ , TOEPH $n$  (among others)
  - *Total* variance and *total* covariance kept in **R**, so **R** = **V**
  - Other than CS, does not partition total variance into BP vs. WP
- **G** and **R** combined models (so **G** and **R** → **V**):
  - Specify random intercept variance  $\tau_{U_0}^2$  in **G** using RANDOM statement, then specify **R** model using REPEATED statement
  - **G** matrix = Level-2 BP variance and covariance due to  $U_{0i}$ , so **R** = Level-1 WP variance and covariance of the  $e_{ti}$  residuals
  - **R** models what's left after accounting for mean differences between persons (via the random intercept variance  $\tau_{U_0}^2$  in **G**)

# Syntax for Models for the Variance

- Does your model include **random intercept variance**  $\tau_{U_0}^2$  (for  $U_{0i}$ ) ?
  - Use the **RANDOM** statement → **G matrix**
  - Random intercept models BP interindividual differences in mean Y
- What about **residual variance**  $\sigma_e^2$  (for  $e_{ti}$ ) ?
  - Use the **REPEATED** statement → **R matrix**
    - **WITHOUT a RANDOM statement: R is BP and WP variance together** =  $\sigma_T^2$   
→ Total variances and covariances (to model all variation, so **R = V**)
    - **WITH a RANDOM statement: R is WP variance only** =  $\sigma_e^2$   
→ Residual variances and covariances to model WP intraindividual variation  
→ **G** and **R** put back together = **V matrix** of total variances and covariances
- The **REPEATED** statement is always there implicitly...
  - Any model **always** has at least one residual variance in **R** matrix
- But the **RANDOM** statement is only there if you write it
  - **G** matrix isn't always necessary (don't always need random intercept)

# Wrapping Up: ACS Models

- Even if you just expect fluctuation over time rather than change, you still should be concerned about accurately predicting the variances and covariances across occasions
- Baseline models (from ANOVA least squares) are CS & UN:
  - Compound Symmetry: Equal variance and covariance over time
  - Unstructured: All variances & covariances estimated separately
  - CS and UN via ML or REML estimation allows missing data
- MLM gives us choices in the middle
  - Goal: Get as close to UN as parsimoniously as possible
  - **R**-only: Structure TOTAL variation in one matrix (**R** only)
  - **G**+**R**: Put constant covariance due to random intercept in **G**, then structural RESIDUAL covariance in **R** (so that **G** and **R** → **V** TOTAL)