An Introduction to Multilevel Models

PSYC 943 (930): Fundamentals of Multivariate Modeling

Lecture 25: December 7, 2012

Today’s Class

- Concepts in Longitudinal Modeling
- Between-Person vs. +Within-Person Models
- Repeated Measures ANOVA as MLM
- Introduction to Multilevel Models
- Fixed and Random Effects of Time and Persons
- Families of Models for Change

Dimensions for Organizing Models

- <u>What kind of outcome variable?</u>
  - Normal/continuous? → "general linear models (GLM)"
  - Non-normal/categorical? → "genera*lized* linear models"
- <u>What kind of predictors?</u> (names relevant only within GLM)
  - Continuous predictors? → "Regression"
  - Categorical predictors? → "ANOVA"
- <u>How many dimensions of sampling are in your data?</u>
  - How many ways do your observations differ from each other?
  - What kind of "**dependency**" or "**correlation**" is in your data?

Multilevel models are used to quantify and predict dependency in an outcome related to different dimensions of sampling.

What is a Multilevel Model (MLM)?

- Same as other terms you have heard of:
  - **General Linear Mixed Model** (if you are from statistics)
    - *Mixed* = Fixed and Random effects
  - **Random Coefficients Model** (also if you are from statistics)
    - Random coefficients = Random effects
  - **Hierarchical Linear Model** (if you are from education)
    - Not the same as hierarchical regression

- Special cases of MLM:
  - Random Effects ANOVA or Repeated Measures ANOVA
  - (Latent) Growth Curve Model (where "latent" → SEM)
  - Within-Person Variation Model (e.g., for daily diary data)
  - Clustered/Nested Observations Model (e.g., for kids in schools)
  - Cross-Classified Models (e.g., "value-added" models)

The Two Sides of a Model

- **Model for the Means:**
  - *Aka* **Fixed Effects**, Structural Part of Model
  - What you are used to **caring about for testing hypotheses**
  - How the expected outcome for a given observation varies as a function of values on predictor variables
  - *Fixed effects are always specified as a function of known things*

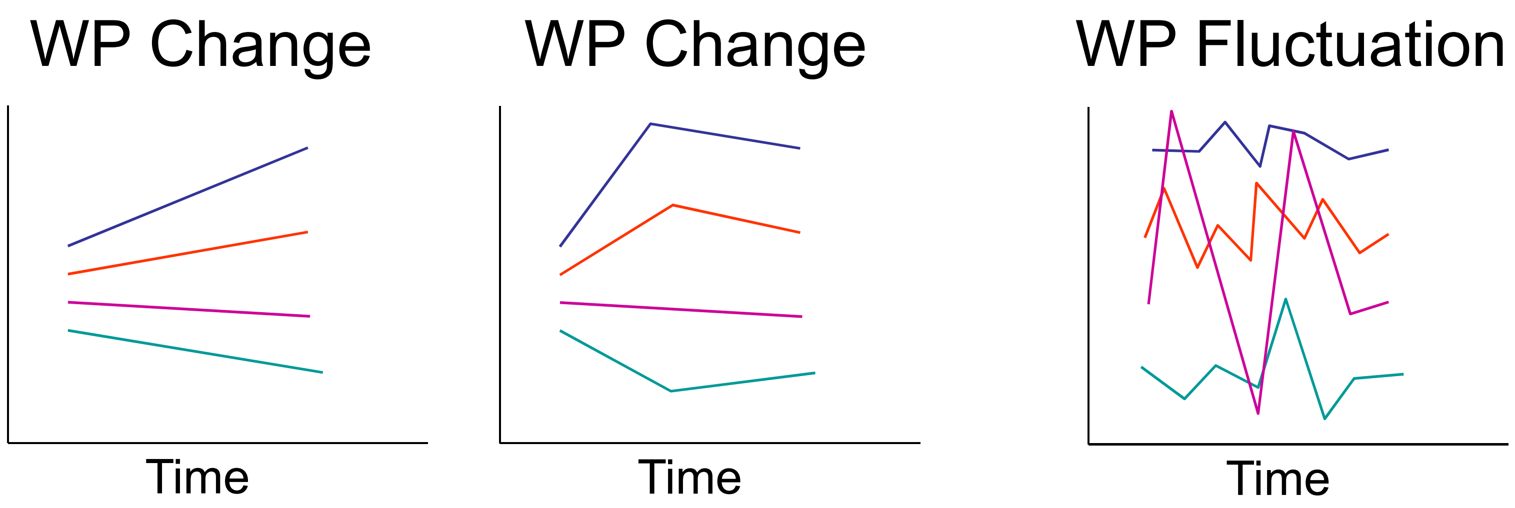
- **Model for the Variances ("piles of variance"):**
  - *Aka* **Random Effects and Residuals**, Stochastic Part of Model
  - What you are used to **making assumptions about** instead
  - Terms that allow model residuals to be related across observations (persons, groups, time, etc) → operate as a function of **sampling**
  - *Random effects and residuals are unknown things ("error")*
  - **This is the primary way that MLM differs from the GLM**

Data Requirements for Longitudinal Modeling

- Multiple measures from the same person!
  - 2 is minimum, but just 2 can lead to problems:
    - Only 1 kind of change is observable (1 difference)
    - Can't directly model interindividual differences in change, because no differentiation between real change and measurement error is possible
  - More data is better (with diminishing returns)
    - More occasions → better description of the form of change
    - More people → better estimates of individual differences in change; better prediction of those individual differences

2 Types of Within-Person Variation

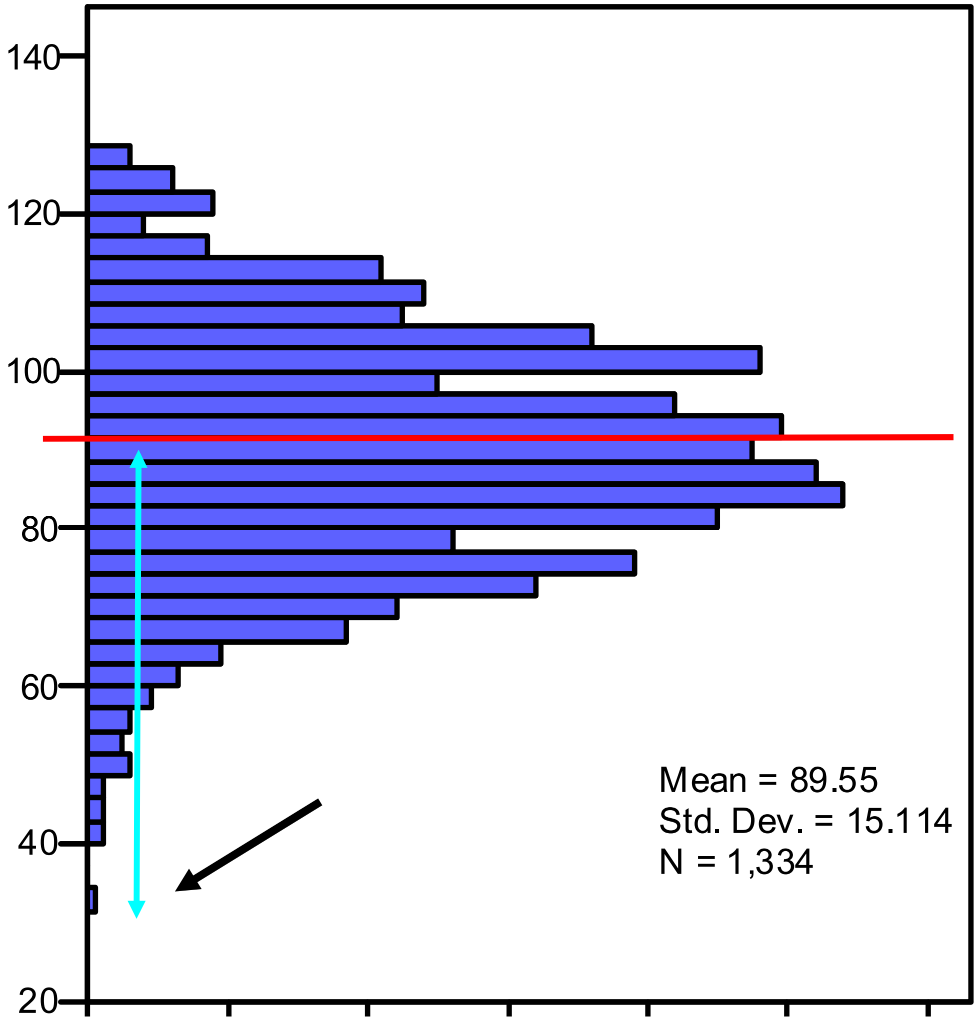
- **Within-Person Change**: Systematic change
  - Magnitude or direction of change can be different across people
  - “Growth curve models” → Time is meaningfully sampled
- **Within-Person Fluctuation**: No systematic change
  - Outcome just varies/fluctuates over time (e.g., emotion, stress)
  - Time is just a way to get lots of data per person



Levels of Inference in Longitudinal Multilevel Modeling

- Between-Person (BP) Relationships:
  - **Level-2** – "**INTER**-individual Differences" – Time-Invariant
  - All longitudinal studies begin as cross-sectional studies
- Within-Person (WP) Relationships:
  - **Level-1** – "**INTRA**-individual Variation" – Time-Varying
  - Only longitudinal studies can provide this extra information
- Longitudinal studies allow examination of both types of relationships simultaneously (and their interactions)
  - Any variable measured over time usually has both of these sources of variation: BP and WP

An Empty Between-Person Model



$$Y_i = \beta_0 + e_i$$

Filling in values:

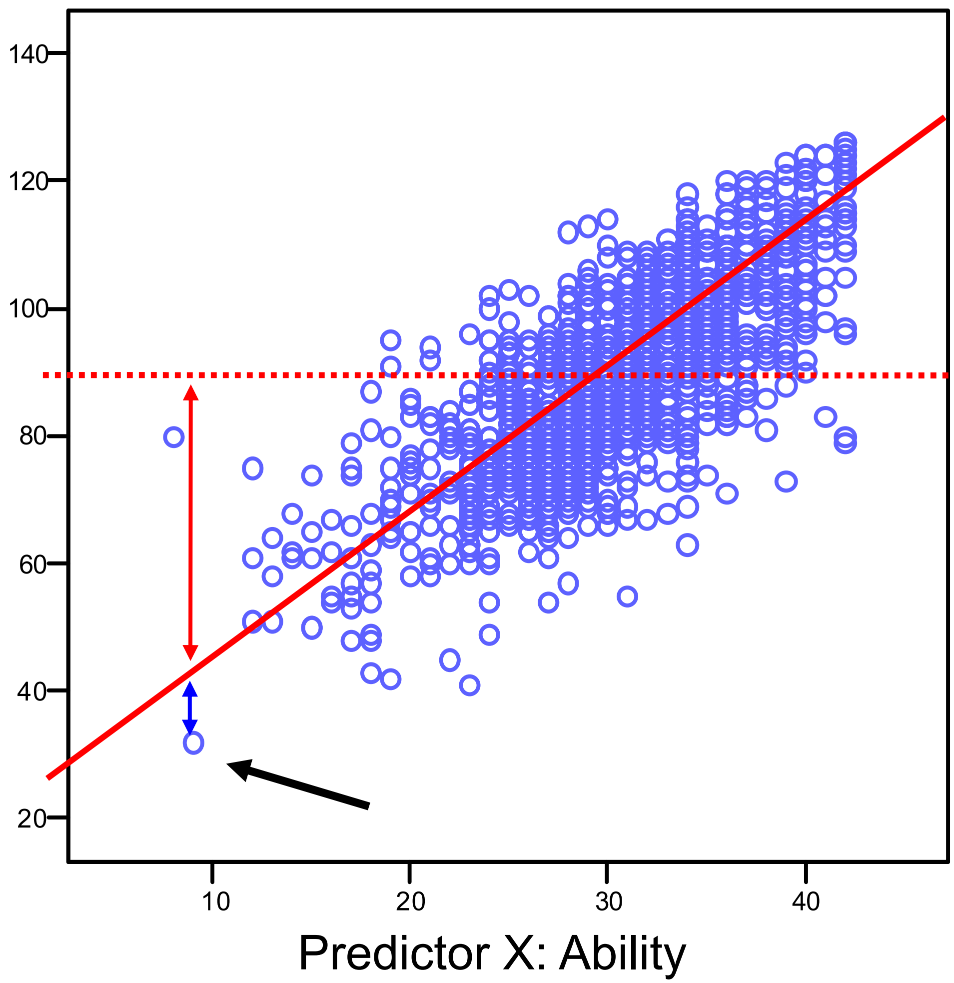
$$32 = \underbrace{90}_{\text{Y pred}} + -58$$

Model for the Means

Y Error Variance:

$$\frac{\Sigma (y - y_{pred})^2}{N - 1}$$

Between-Person Model: One Continuous Predictor



$$y_i = \beta_0 + \beta_1 X_i + e_i$$

Empty Model:

$32 = 90 + \quad -58$

Ability (X) Model:

$32 = 29 + 2*9 + -15$

Y pred

Model for the Means

Y Error Variance:

$$\frac{\Sigma (y - y_{pred})^2}{N - 2}$$

A More General Linear Model for Between-Person Analysis

$$y_i = \boxed{\beta_0 + \beta_1 X_i + \beta_2 Z_i + \beta_3 X_i Z_i} + e_i$$

Model for the Means (Fixed Effects):

- Each person's expected (predicted) outcome is a function of his/her values on x and z (and here, of their interaction, too)

- Even though the grand mean is no longer the best guess (best model) for each person's outcome, we still call it "the model for the means" because each person gets a **conditional mean** as his or her best guess (same predicted Y given same predictor values)

A More General Linear Model for Between-Person Analysis

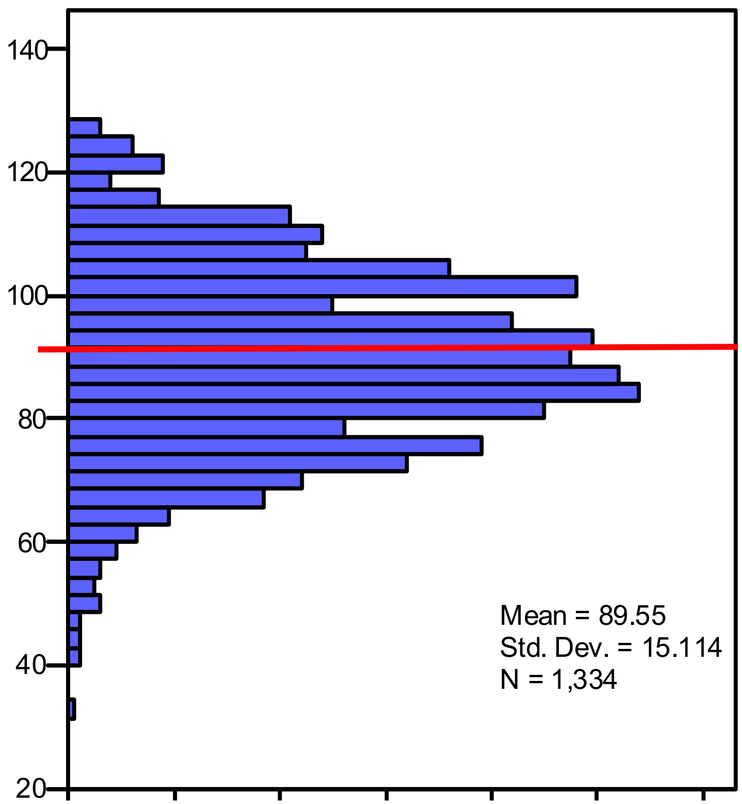
$$y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + \beta_3 X_i Z_i + \boxed{e_i}$$

Model for the Variance (Residuals and Random Effects):

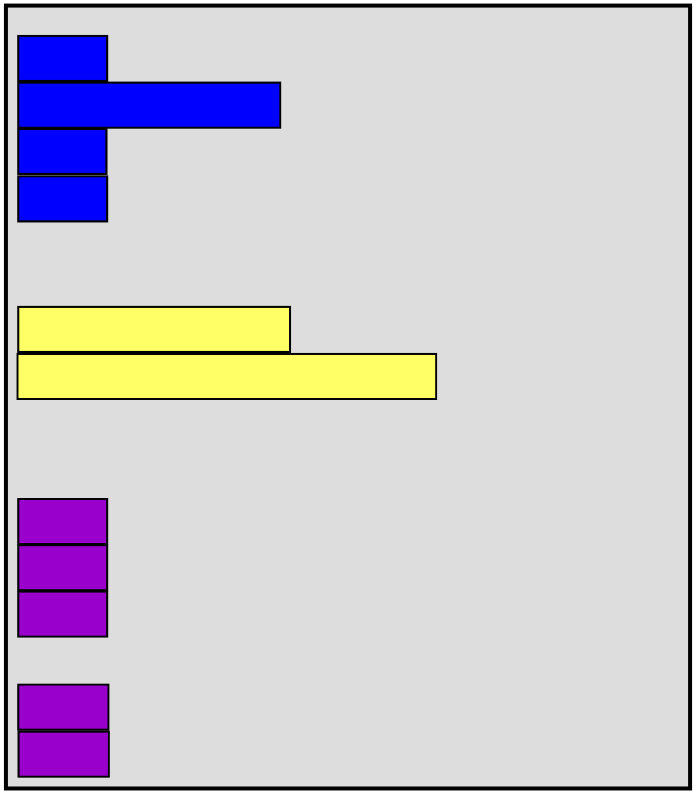
- $e_i \sim NID(0, \sigma_e^2)$ → In English: e_i is a random variable with a mean of 0 and some estimated variance, and is normally distributed
- **ONE** pile of variance in leftover Y after accounting for predictors
- We don’t care about the individual e’s – we care about their variance
- What makes this model for the variance “Between-Person”?
  - e_i is measured only once per person (as indicated by the *i* subscript)

Adding Within-Person Variance to the Model for the Variances

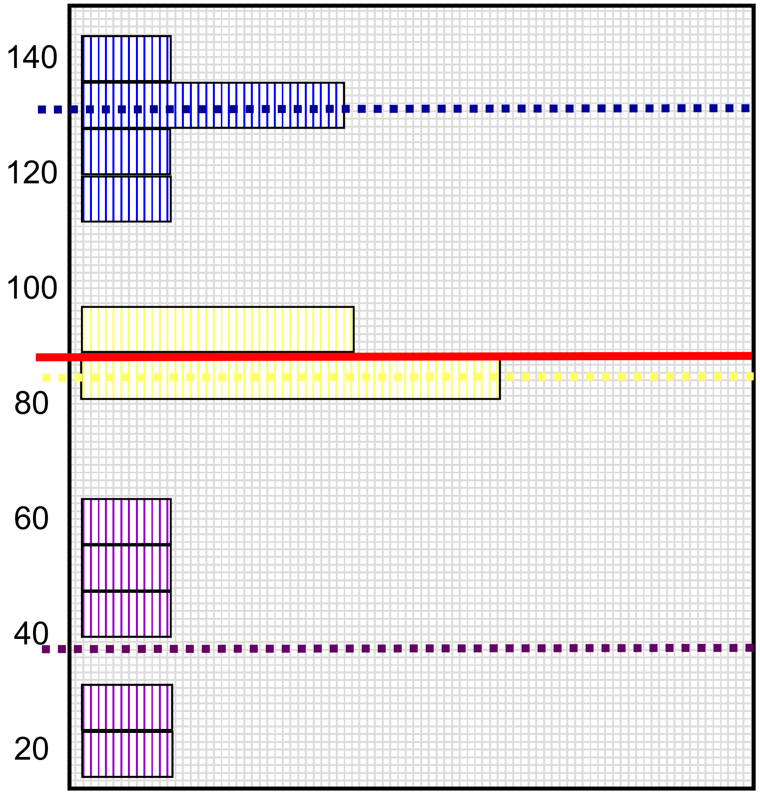
Full Sample Distribution:



3 People, 5 Occasions each



Empty +Within-Person Model



Start off with Mean of Y as "best guess" for any value:

= Grand Mean

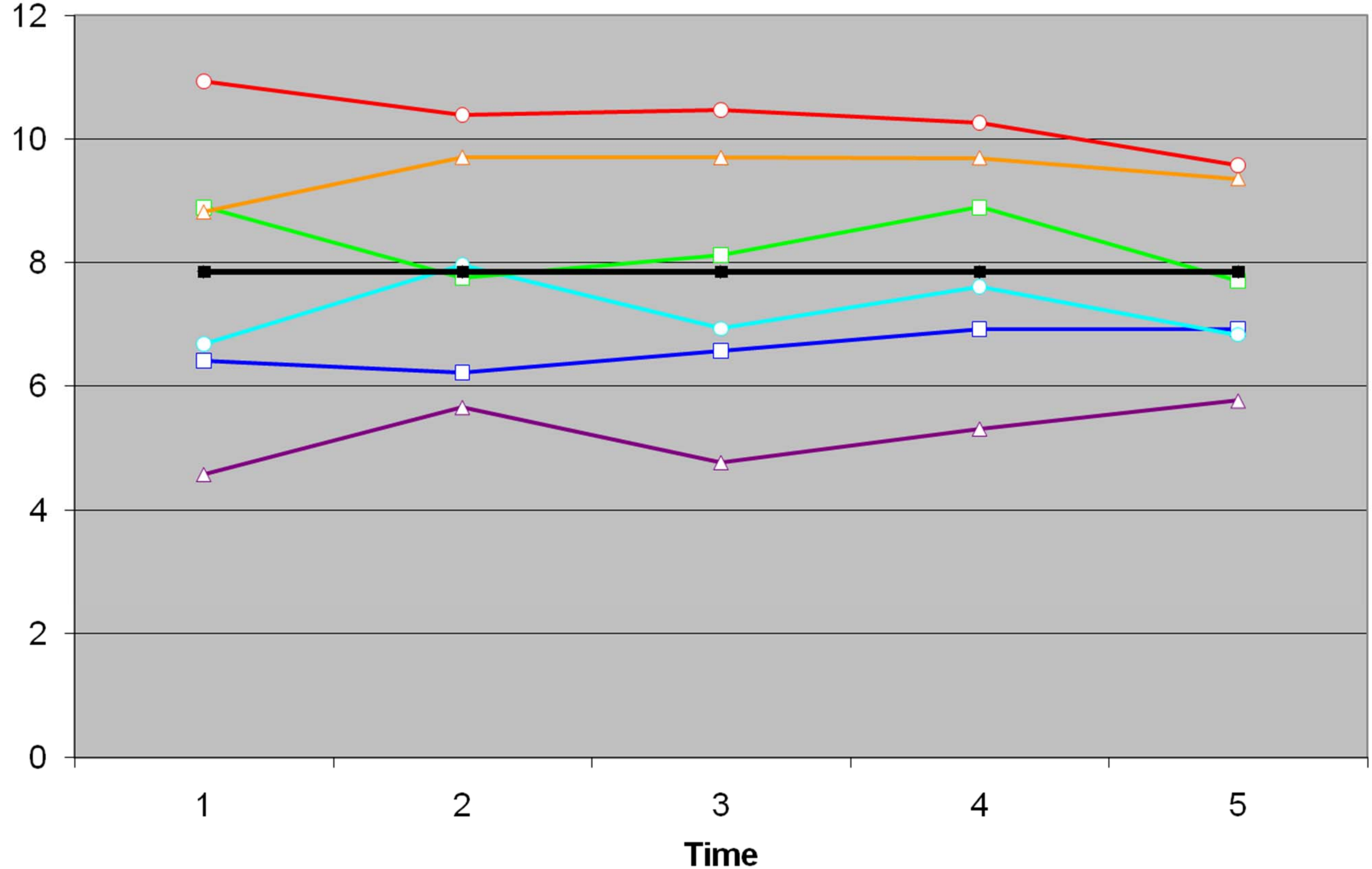
= Fixed Intercept

Can make better guess by taking advantage of repeated observations:

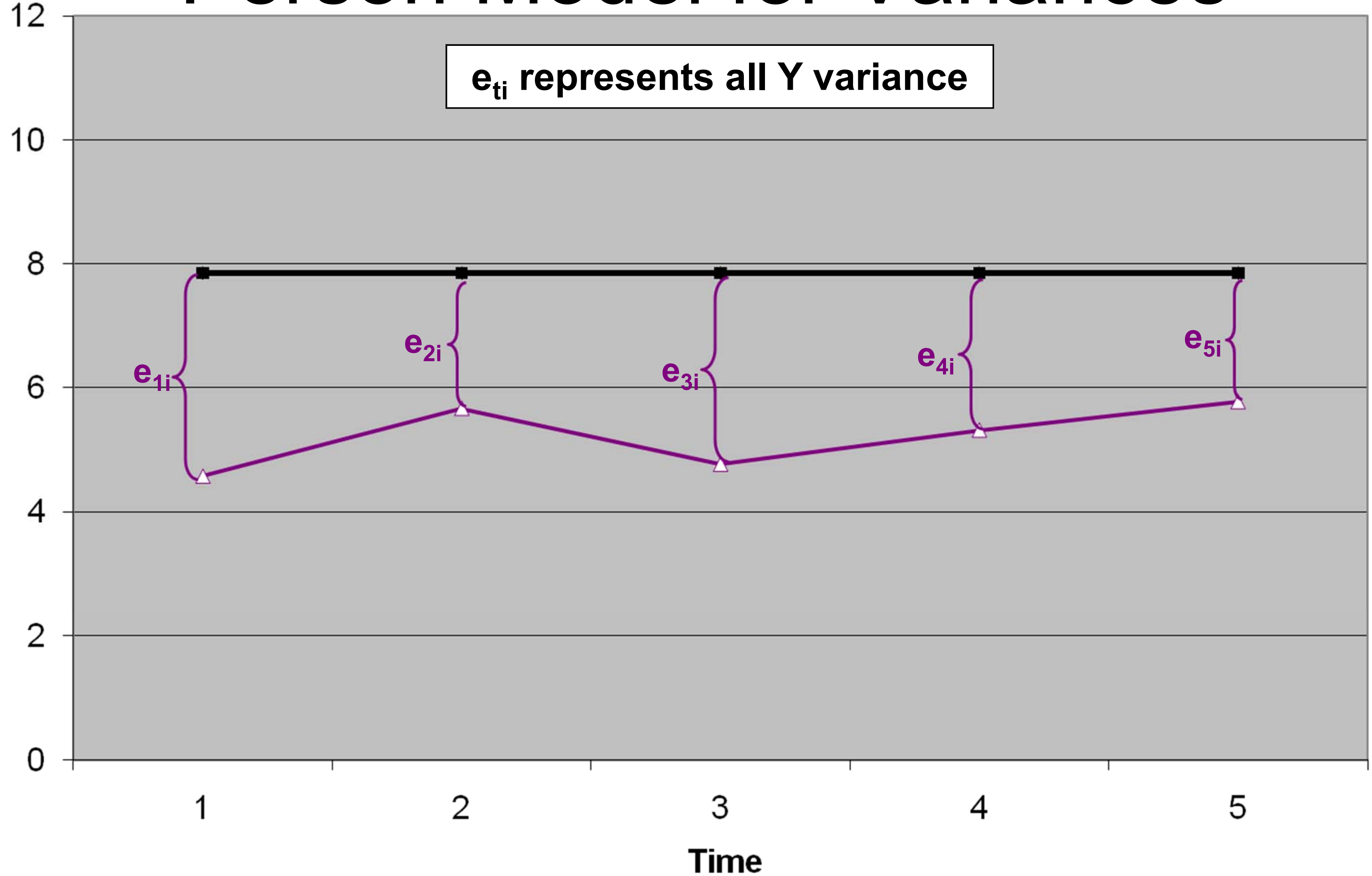
= Person Mean

→ Random Intercept

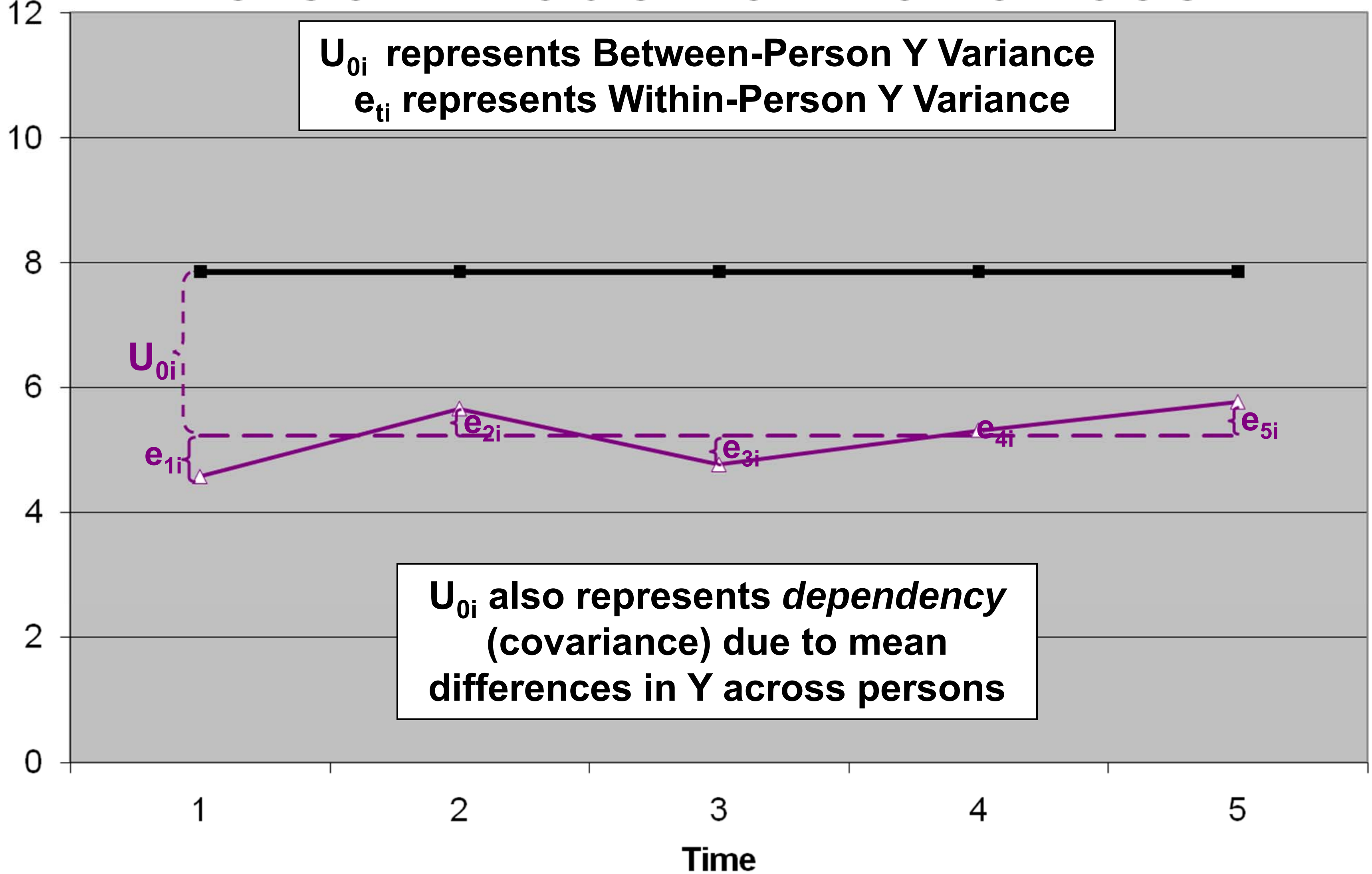
Division of Error in **Between** Person Model for Variances



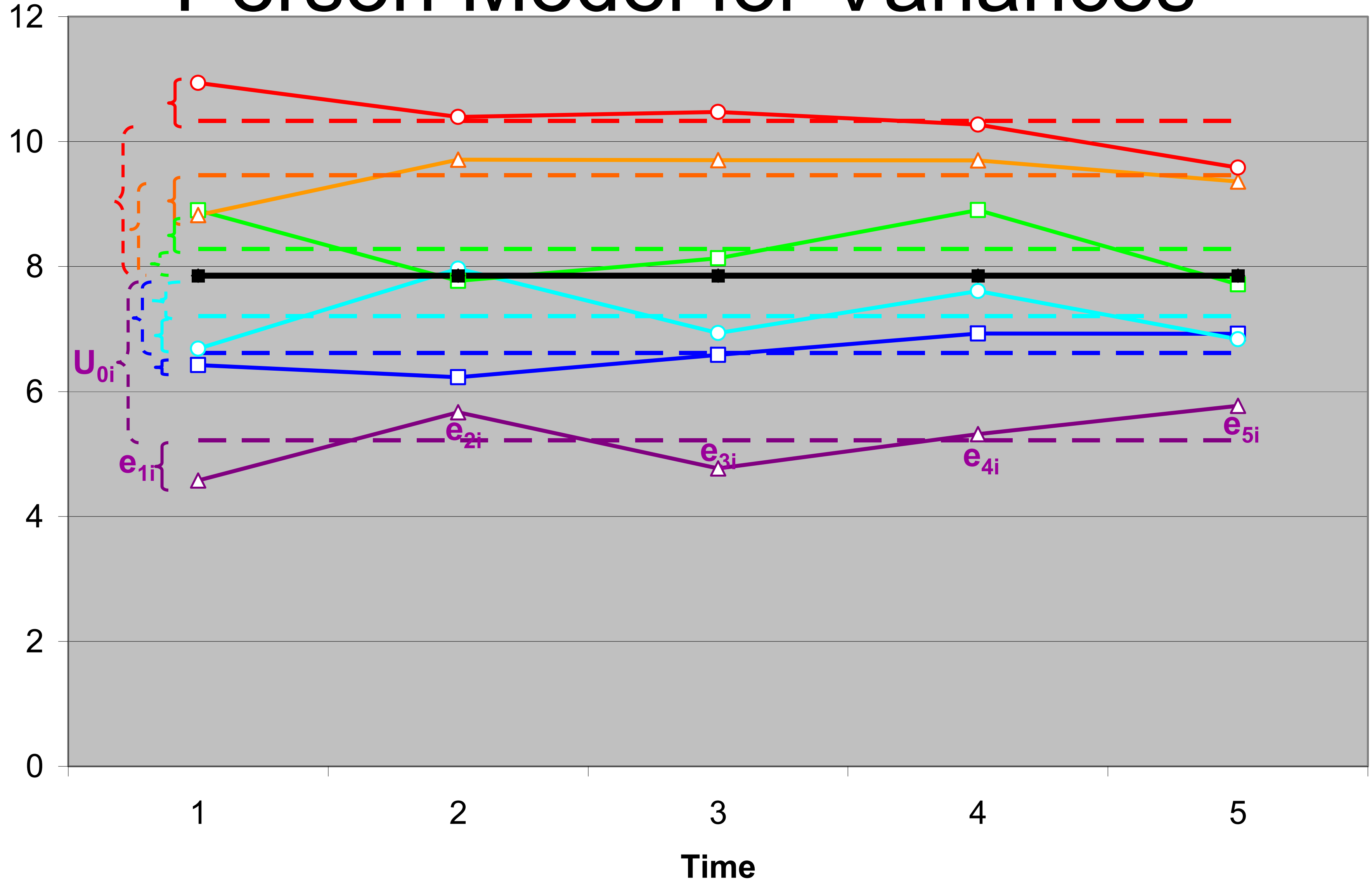
Division of Error in **Between** Person Model for Variances



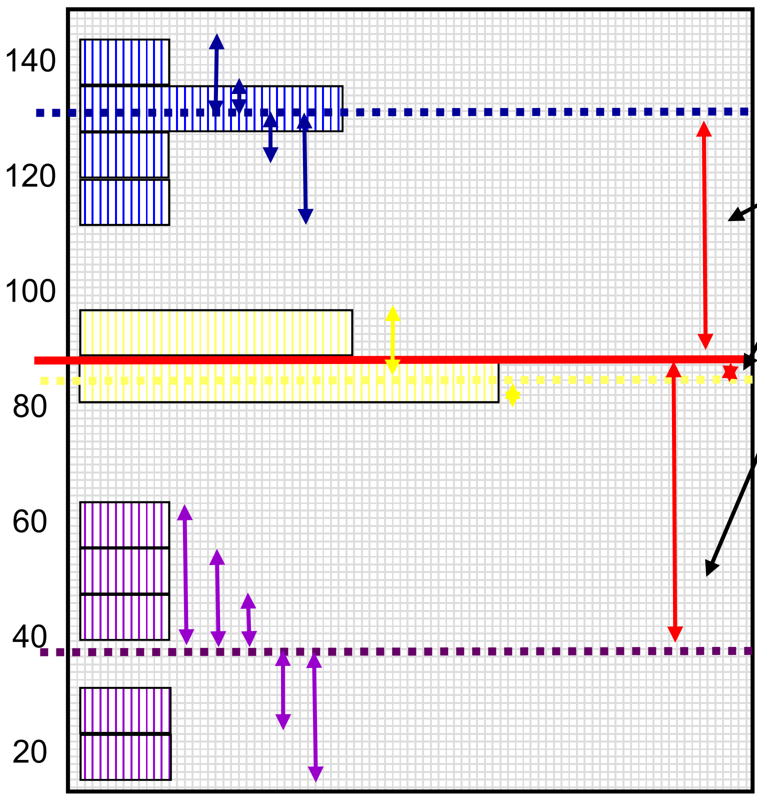
Division of Error in **+Within** Person Model for Variances



Division of Error in **+Within** Person Model for Variances



Empty +Within-Person Model



Variance of Y → 2 sources:

Between-Person Variance:

→ Differences from GRAND mean

→ INTER-Individual Differences

Within-Person Variance:

→ Differences from OWN mean

→ INTRA-Individual Differences

Now we have 2 piles of variance in Y to predict.

Between-Person vs. +Within-Person Empty Models

- Empty Between-Person Model (1 time point):

$$y_i = \beta_0 + e_i$$

  - β_0 = fixed intercept = grand mean
  - e_i = residual deviation from GRAND mean

- Empty +Within-Person Model (>1 time points):

$$y_{ti} = \beta_0 + U_{0i} + e_{ti}$$

  - β_0 = fixed intercept = grand mean
  - U_{0i} = random intercept = individual deviation from GRAND mean
  - e_{ti} = time-specific residual deviation from OWN mean

The Model for the Variances

- All models have at least an “e” → 1 pile of variance
- 2 main reasons to care about *what else* should go into the **model for the variances**:
  - **Validity of the tests of the predictors** depends on having the ‘right’ model for the variances (where ‘right’ means ‘least wrong’)
    - Estimates will usually be ok → come from model for the means
    - Standard errors (and thus p-values) can be compromised
  - The sources of variation that exist in your outcome will dictate **what kinds of predictors** will be useful
  - For example, in longitudinal data:
    - Between-Person variation needs Between-Person predictors
    - Within-Person variation needs Within-Person predictors

Categorizing Familiar Models by Their Models for the Variances

- Multiple Regression, Between-Person ANOVA: **1 PILE**
  - $\mathbf{y_i = (\beta_0 + \beta_1 X_i + \beta_2 Z_i \ldots) + e_i}$
  - $\mathbf{e_i}$ → ONE residual, assumed uncorrelated with equal variance across observations (here, just persons) → "**BP variation**"

- Repeated Measures, Within-Person ANOVA: **2 PILES**
  - $\mathbf{y_{ti} = (\beta_0 + \beta_1 X_i + \beta_2 Z_i \ldots) + U_{0i} + e_{ti}}$
  - $\mathbf{U_{0i}}$ → A random intercept for differences in person means, assumed uncorrelated with equal variance across persons → "**BP variation**" = $\tau^2_{U_0}$
  - $\mathbf{e_{ti}}$ → A residual that represents remaining time-to-time variation, usually assumed uncorrelated with equal variance across observations (now, persons and time) → "**WP variation**" = σ^2_e

The Model for the Variances

- All models have at least an "e" → 1 pile of variance
- 2 main reasons to care about *what else* should go into the **model for the variances**:
  - **Validity of the tests of the predictors** depends on having the "right" model for the variances (where "right" means "least wrong")
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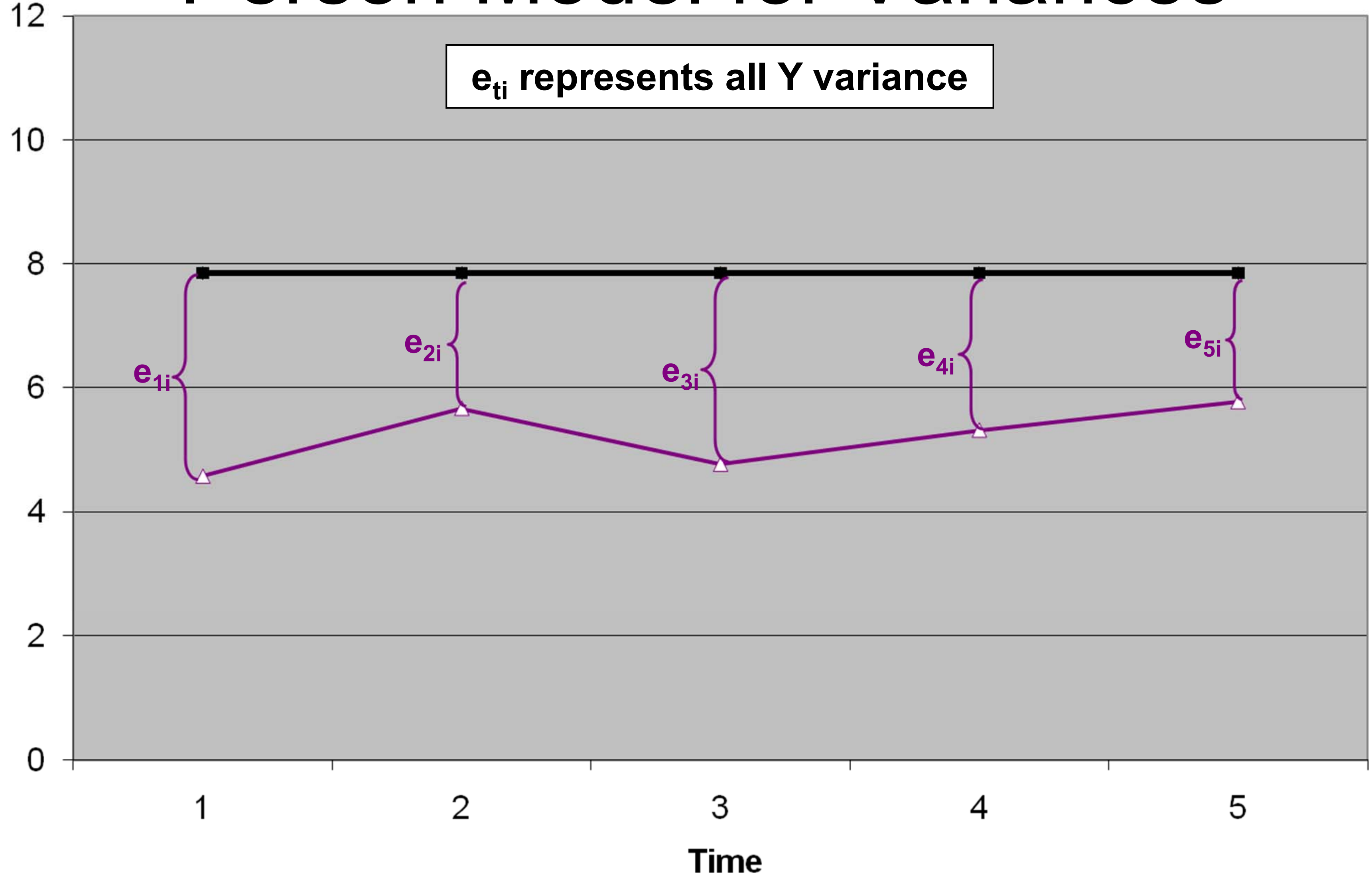
  - For example, in longitudinal data:
    - Between-Person variation needs Between-Person predictors
    - Within-Person variation needs Within-Person predictors

ANOVA for longitudinal data?

- There are 3 possible "kinds" of ANOVAs we could use:
  - Between-Persons/Groups, Univariate RM, and Multivariate RM
- **NONE OF THEM ALLOW:**
  - **Missing occasions (do listwise deletion instead)**
  - **Time-varying predictors**
- Each includes the same model for the means with respect to time: all possible mean differences (so 4 parameters to get to 4 means)
  - **"Saturated means model":** $\beta_0 + \beta_1(T_1) + \beta_2(T_2) + \beta_3(T_3)$
  - **The *Time* variable must be balanced and discrete in ANOVA!**
- Each kind of ANOVA differs by what it says about the correlation in the data from the same person in the model for the variances…
  - i.e., how it "handles dependency" due to persons, or what it says the variance and covariance of the y_{ti} residuals should look like…

Division of Error in **Between** Person Model for Variances



ANOVA for longitudinal data?

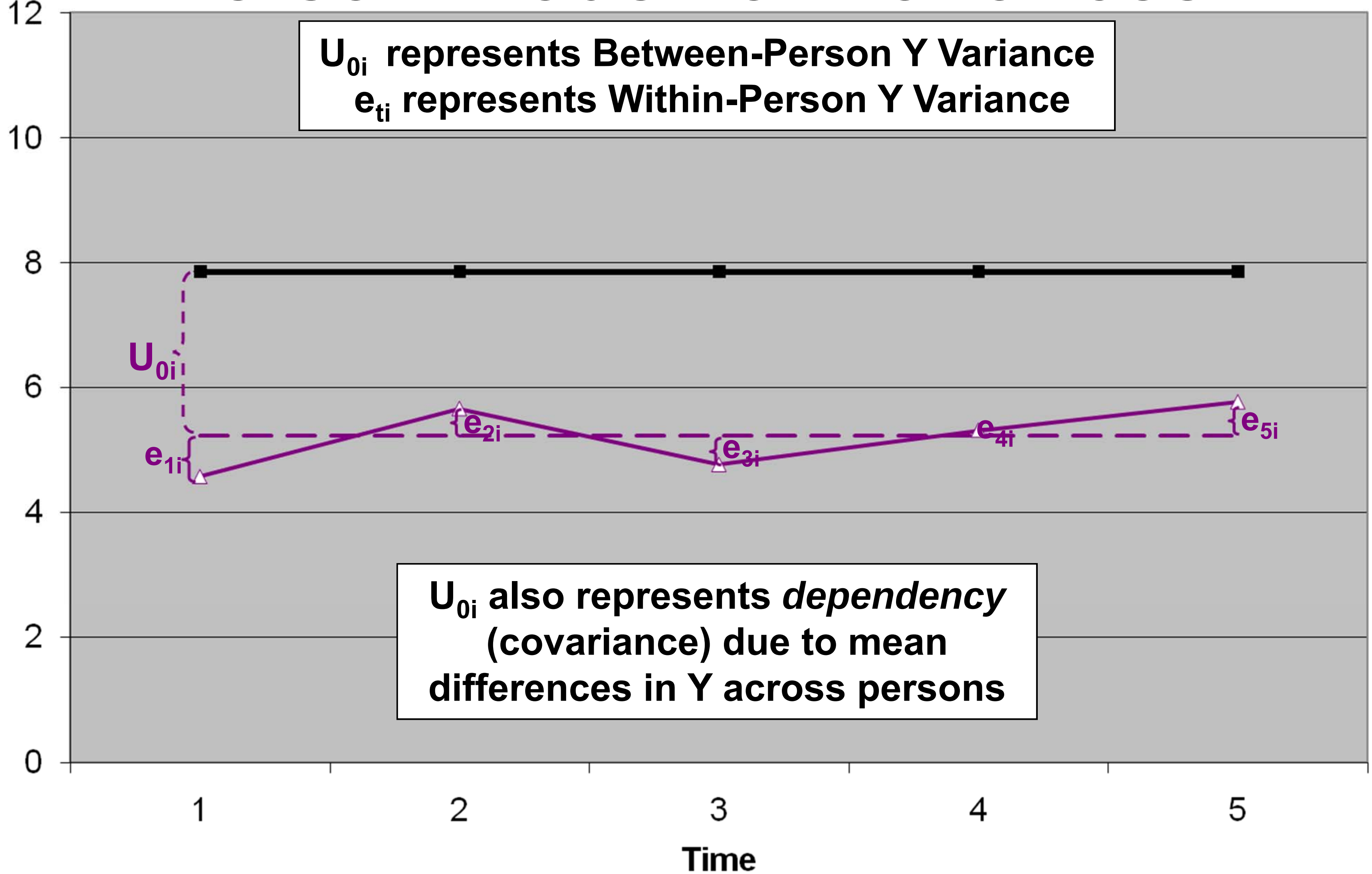
1. **Between-Groups Regression or ANOVA**
   – BP variance only (1 pile of e_{ti} only)

$$\begin{bmatrix} \sigma_e^2 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 \\ 0 & 0 & \sigma_e^2 & 0 \\ 0 & 0 & 0 & \sigma_e^2 \end{bmatrix}$$

   – Assumes NO RELATIONSHIP WHATSOEVER among observations from the same person (or across persons)
      - **Dependency? What dependency?**
      - e.g., 4 occasions * 100 persons would be 400 "independent observations"

   – Will usually be VERY WRONG for longitudinal data
      - BP effects tested against wrong df, WP effects tested against wrong df and wrong variance → messed up SEs → messed up *p*-values
      - Will also be wrong for clustered data, although perhaps less so *(because the correlation among persons from the same group in clustered data is not as strong as the correlation among occasions from the same person in longitudinal data)*

Division of Error in **+Within** Person Model for Variances



ANOVA for longitudinal data?

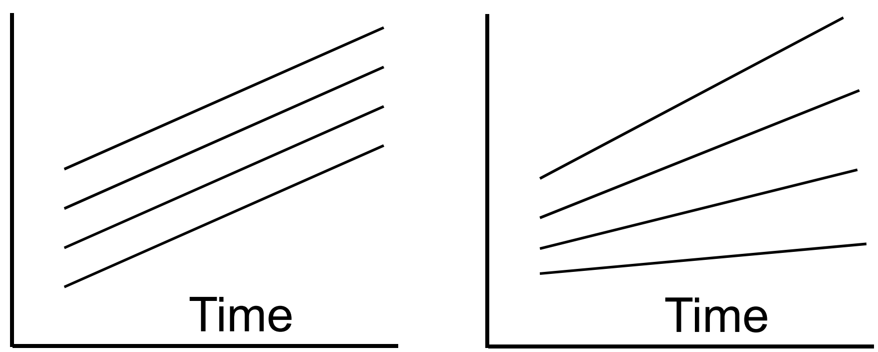
2. (a) Univariate Repeated Measures ANOVA: Var(e_{ti}) + Var(U_{0i})

- Assumes a **CONSTANT RELATIONSHIP OVER TIME** among observations from the same person: Compound Symmetry

Observations from the same person are correlated because of constant person mean differences (via U_{0i})
→ **only 1 kind of person dependency**

$$\begin{bmatrix} \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 \end{bmatrix}$$

- Will usually be SOMEWHAT WRONG for longitudinal data
  - **If people change at different rates, the variances and covariances of the outcome over time have to change, too**



ANOVA for longitudinal data?

2. (b) Univariate RM ANOVA with sphericity corrections

- Based on ε → how far off sphericity, (ranges 0-1, 1=spherical)
- Applies an overall correction for model df based on estimated ε
- Corrections for sphericity do not really solve the problem

3. Multivariate Repeated Measures ANOVA

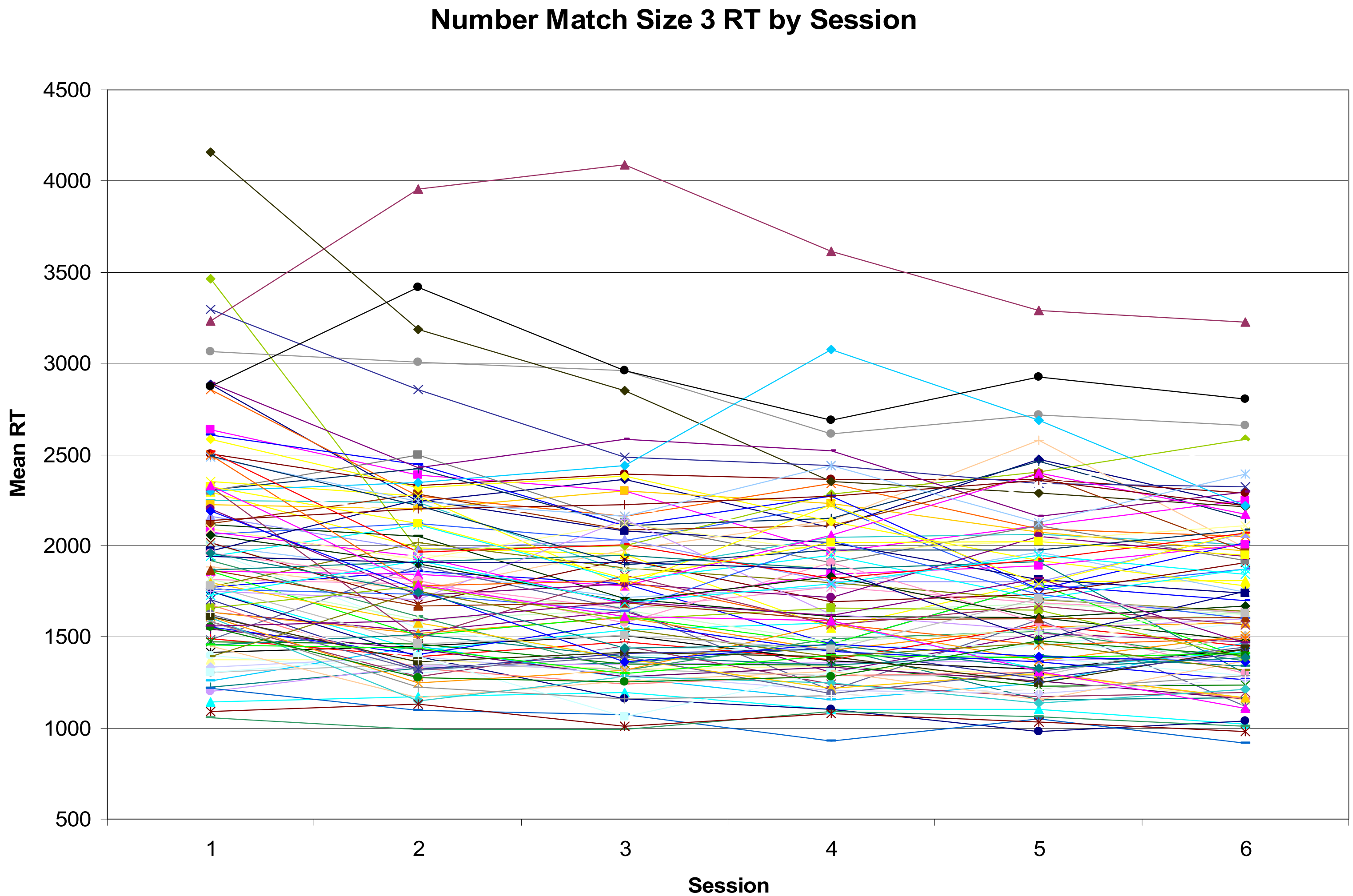
- Assumes nothing: all variances and covariances are estimated separately → "**Unstructured**"

It's not a model, it IS the data!

$$\begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22}^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}^2 & \sigma_{43} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}^2 \end{bmatrix}$$

- Because it can never it be wrong, an unstructured model can be useful for *complete* longitudinal data with few occasions
- Becomes hard to estimate very quickly with many occasions
  - Parameters needed = (#occasions * [#occasions+1]) / 2

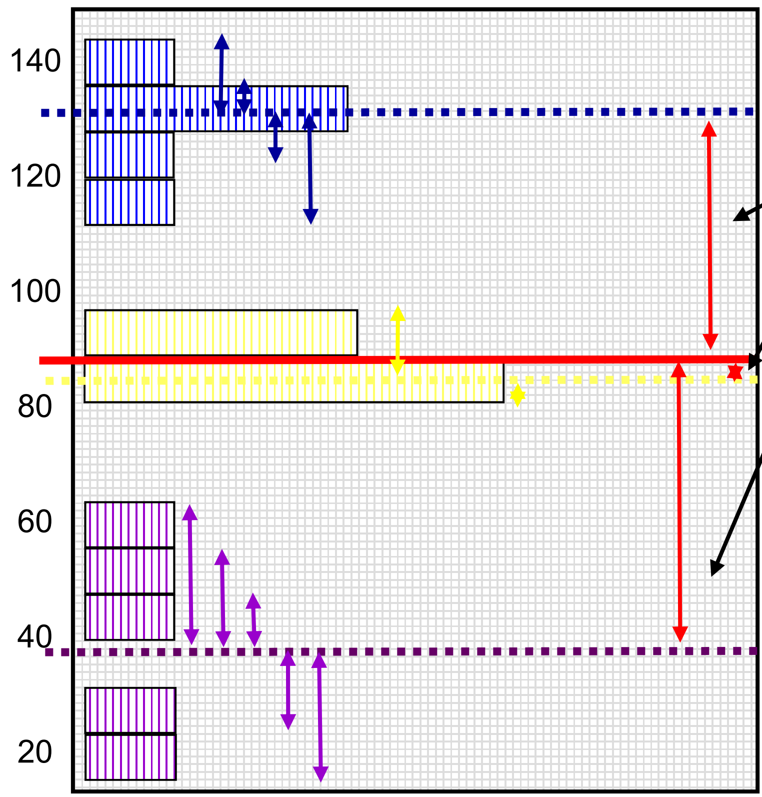
Example Data Individual Observed Trajectories (N = 101, n = 6)



Summary: ANOVA approaches for longitudinal data are "one size fits most":

- **Saturated Model for the Means** (balanced time required)
  - All possible mean differences
  - Unparsimonious, but best-fitting (is a description, not a model)
- **3 kinds of Models for the Variances** (complete data required)
  - e_{ti} only = Between-Person/Group ANOVA → assumes independent data
  - U_{0i} and e_{ti} = Compound Symmetry (CS) = Univ. RM ANOVA
    - Requires sphericity, which rarely holds in longitudinal data
  - All possible var. and covar. = Unstructured (UN) = Multiv. RM ANOVA
    - Unparsimonious; is a description, not a model
- **MLM will give us more flexibility in both parts of the model:**
  - Fixed effects that *predict* the pattern of means
  - Random intercepts and slopes and/or alternative covariance structures that *predict* the pattern of variation and covariation over time

Empty Longitudinal Multilevel Model: Review of Terminology



Variance of Y → 2 sources:

<u>Level 2 Random Intercept Variance</u> (of U_{0i}):

- → Between-Person Variance (τ_{U0}^2)
- → Difference from GRAND mean
- → INTER-Individual Differences

<u>Level 1 Residual Variance</u> (of e_{ti}):

- → Within-Person Variance (σ_e^2)
- → Difference from OWN mean
- → INTRA-Individual Differences

Empty* Multilevel Model

Model for the Means; Model for the Variance

General Linear Model

$$y_i = \beta_0 + e_i$$

Multilevel Model

Level 1: $y_{ti} = \beta_{0i} + e_{ti}$

Level 2: $\beta_{0i} = \gamma_{00} + U_{0i}$

Sample **Grand Mean Intercept** (→ γ_{00})

Individual **Intercept Deviation** (→ U_{0i})

3 Model Parameters

1 Fixed Effect:

γ_{00} → fixed intercept

1 Random Effect (intercept):

U_{0i} → person-specific deviation

→ mean=0, variance = τ_{U0}^2

1 Residual Error:

e_{ti} → time-specific deviation

→ mean=0, variance = σ_e^2

Composite equation: $y_{ti} = \gamma_{00} + U_{0i} + e_{ti}$

* To be more clear, I call this an "empty means, random intercept" model

Empty Multilevel Model: Useful Descriptive Statistic → ICC

Intraclass Correlation (ICC):

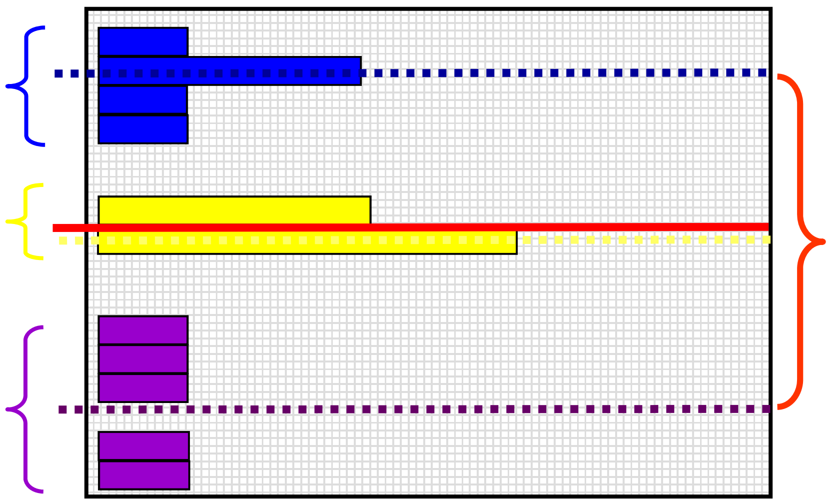
$$\text{ICC} = \frac{\text{Intercept Variance}}{\text{Intercept Variance + Residual Variance}} = \frac{\tau_{U_0}^2}{\tau_{U_0}^2 + \sigma_e^2}$$

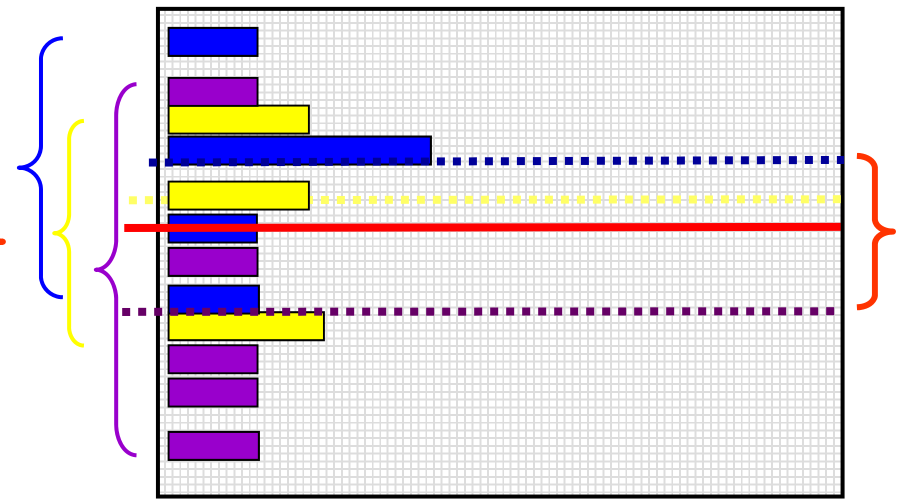
$$\text{ICC} = \frac{\text{Between-Person Variance}}{\text{Between-Person Variance + Within-Person Variance}}$$

- ICC = Proportion of total variance that is between persons
- ICC = Average correlation among occasions
- ICC is a standardized way of expressing how much we need to worry about *dependency due to person mean differences* **(i.e., ICC is an effect size for *constant* person dependency)**

$$ICC = \frac{\text{Between-Person Variance}}{\text{Between-Person Variance + Within-Person Variance}}$$

<u>Counter-Intuitive:</u> Between-Person Variance is in the numerator, but the ICC is the correlation over time!





<u>ICC</u> = BTW / BTW + within

→ Large ICC

→ Large correlation over time

<u>ICC</u> = btw / btw + WITHIN

→ Small ICC

→ Small correlation over time

More New Vocabulary: Labels for 3 Types of Models

- "Empty Model" → "Empty means, random intercept model"
  - Just fixed intercept, random intercept variance, residual variance
  - First baseline model for everything to follow
  - Used to compute an ICC

- **"Unconditional (Growth) Model" is up next**
  - Effects related to time, but no other predictors yet
  - Second baseline model for everything to follow

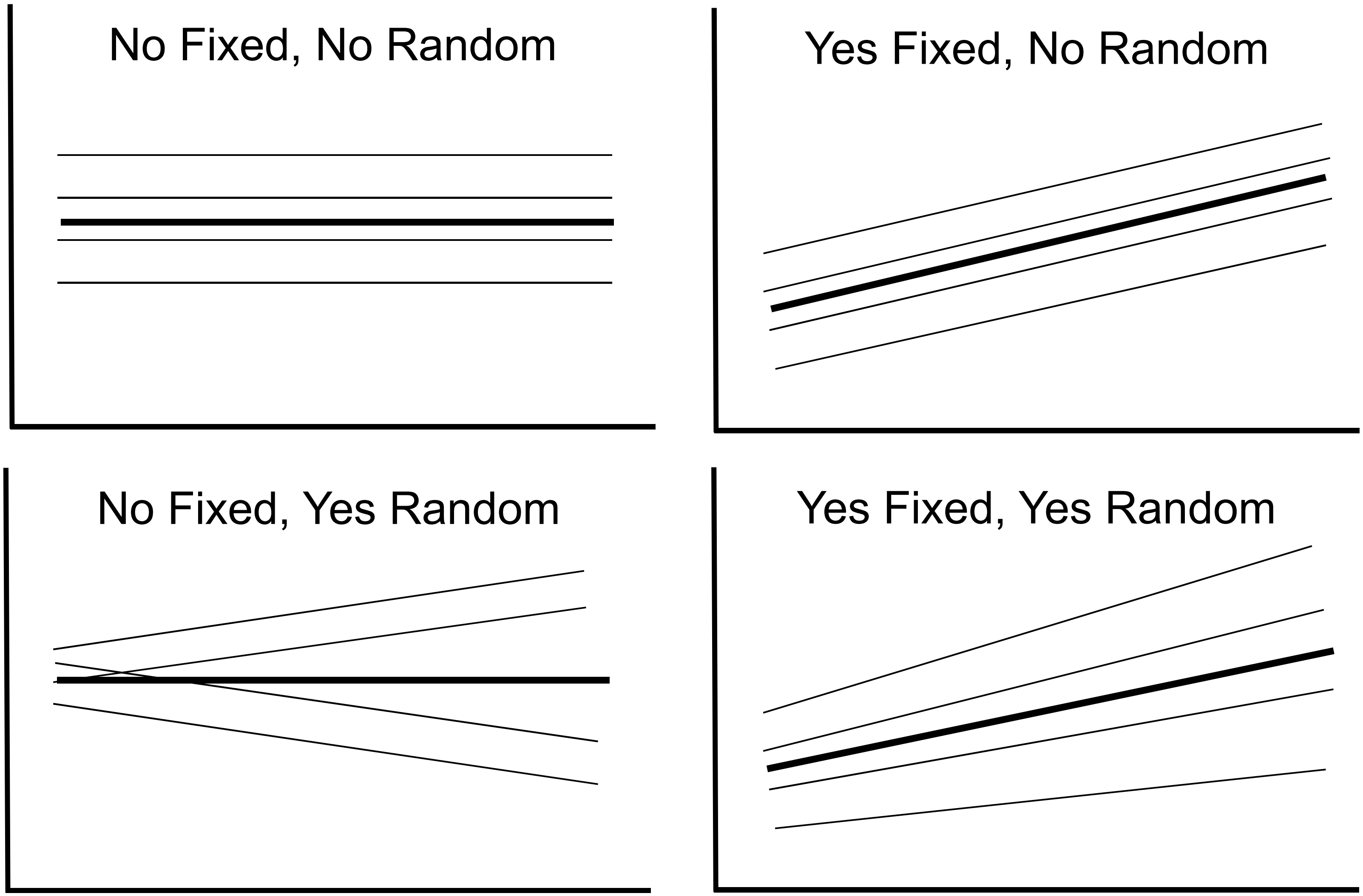
- "Conditional (Growth) Model" is coming next semester
  - With predictors besides time
  - May end up with more than 1 depending on research questions

Extending the Multilevel Model: 2 Questions for Effects of Time

1. Is there an effect of time **on average**?
   → Is the average line not flat?
   → Significant **Fixed** Effect of Time

2. Does the average effect of time **vary across individuals**?
   → Does each person need their own line?
   → Significant **Random** Effect of Time

Fixed and Random Effects of Time



Random Linear Time Model

6 Model Parameters

2 Fixed Effects: γ_{00} and γ_{10}

2 Random Effects (+1 covariance):

Variances of U_{0i} and U_{1i} (τ_{U0}^2, τ_{U1}^2)

Covariance of U_{0i} and U_{1i} (τ_{U01})

1 Residual Variance of e_{ti} (σ_e^2)

Multilevel Model

Level 1: $y_{ti} = \beta_{0i} + \beta_{1i}Time_{ti} + e_{ti}$

Sample (Grand Mean) **Intercept** — Individual **Intercept Deviation** — Sample (Grand Mean) **Slope** — Individual **Slope Deviation**

Level 2: $\beta_{0i} = \gamma_{00} + U_{0i}$ $\quad\quad$ $\beta_{1i} = \gamma_{10} + U_{1i}$

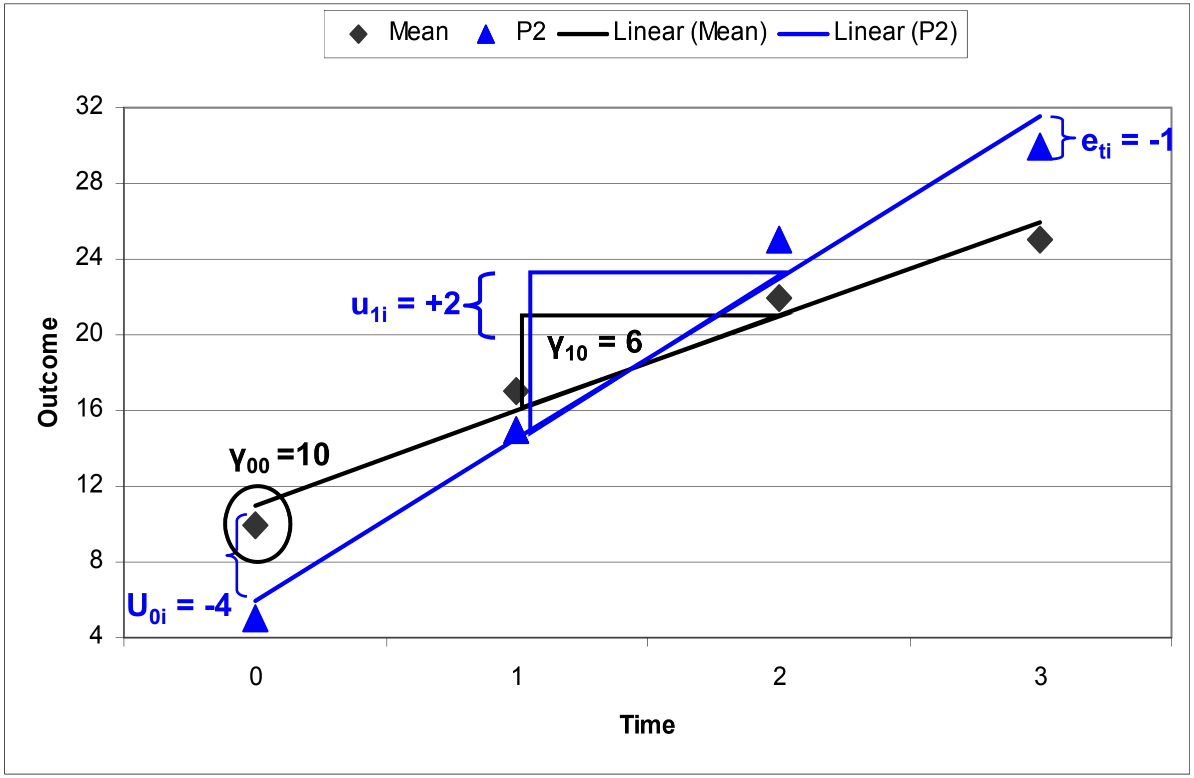
Composite Model

$$y_{ti} = \underbrace{(\gamma_{00} + U_{0i})}_{\beta_{0i}} + \underbrace{(\gamma_{10} + U_{1i})}_{\beta_{1i}}Time_{ti} + e_{ti}$$

Fixed & Random Effects of Time

$$y_{ti} = (\gamma_{00} + U_{0i}) + (\gamma_{10} + U_{1i})Time_{ti} + e_{ti}$$

- γ_{00}: **Fixed Intercept**
- U_{0i}: Random **Intercept** Deviation
- γ_{10}: **Fixed Slope**
- U_{1i}: Random **Slope** Deviation
- e_{ti}: error for person i at time t



6 Model Parameters:

2 Fixed Effects:

γ_{00} Intercept, γ_{10} Slope

2(+1) Random Effects:

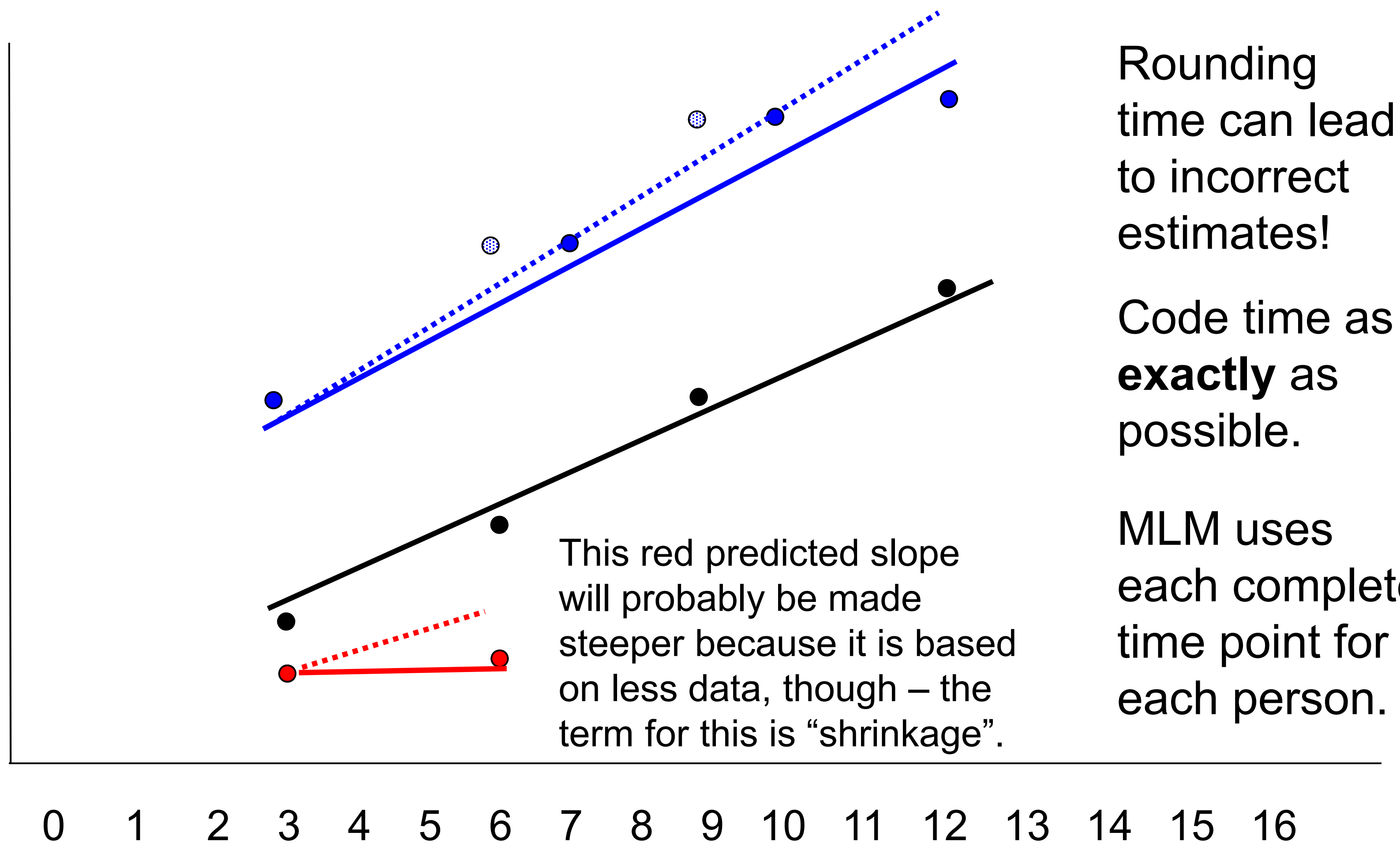
U_{0i} Intercept Variance

U_{1i} Slope Variance

Int-Slope Covariance

1 e_{ti} Residual Variance

Unbalanced Time → Different time measurements across persons? OK



Rounding time can lead to incorrect estimates!

Code time as **exactly** as possible.

MLM uses each complete time point for each person.

Longitudinal Data: Modeling Means and Variances

- We have two tasks in describing the effects of "time":

1. **Choose a Model for the Means**
   - What kind of change in the outcome do we have on average?
   - What kind of and how many parameters do we need to represent that change as parsimoniously but accurately as possible?

2. **Choose a Model for the Variances**
   - What kind of pattern do the variances and covariances of the outcome show over time *due to individual differences in change*?
   - What kind of and how many parameters do we need to represent that pattern as parsimoniously but accurately as possible?

Big Picture Modeling Framework: *Choices for Modeling Means*

- **What kind of change on average (in the means)?**
  - "**Empty**" refers to model for the means with no predictors (just fixed intercept for **grand mean** outcome over time)
  - "**Saturated**" refers to model for means with all possible means estimated (#parameters = #occasions) → **THIS IS ANOVA**
    - Is a DESCRIPTION of the means, not a predictive MODEL

Parsimony

Empty Means:
Predicts NO change over time
1 Fixed Effect

In-between options:
polynomial slopes, piecewise slopes, nonlinear models…

Saturated Means:
Reproduces mean at each occasion
 # Fixed Effects = # Occasions

Good fit

Even if time is unbalanced!

Choices for Modeling Variance

- The partitioning of variance into piles…
  - Level 2 = BP → **G** matrix of random effects variances/covariances
  - Level 1 = WP → **R** matrix of residual variances/covariances
  - **G** and **R** combine to create **V** matrix of total variances/covariances
  - Many flexible options that allows the variances and covariances to vary in a time-dependent way that better matches the actual data

Compound Symmetry (CS)

$$\begin{bmatrix} \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 & \tau_{u_0}^2 \\ \tau_{u_0}^2 & \tau_{u_0}^2 & \tau_{u_0}^2 & \sigma_e^2+\tau_{u_0}^2 \end{bmatrix}$$

MLM uses **G** and **R** to predict something in-between!

Unstructured (UN)

$$\begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22}^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}^2 & \sigma_{43} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}^2 \end{bmatrix}$$

Parsimony ←→ **Good fit**

Univariate RM ANOVA

Even if time is unbalanced!

Multivariate RM ANOVA

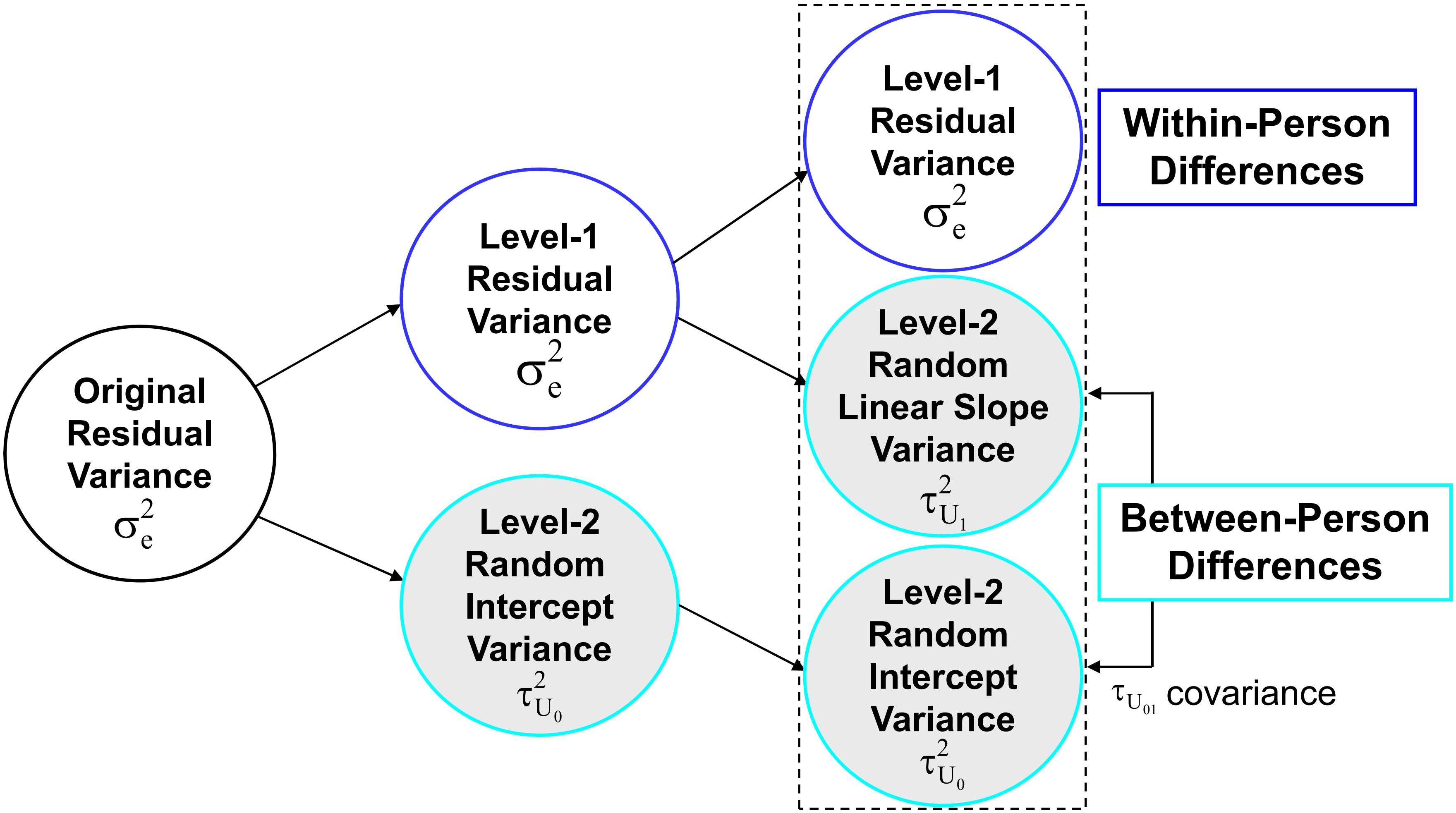
The Point of MLM: Dependency

- Common description of the purpose of MLM is that it 'addresses' or 'handles' correlated (dependent) data…
- But where does this 'correlation' come from?
  3 places (here, an example with health as an outcome):

1. *Mean differences across persons*
   - Some people are just healthier than others (at every time point)
2. *Differences in effects of predictors across persons*
   - Does *time* affect health more in some persons than others?
   - Does *daily stress* affect health more in some persons than others?
3. Non-constant within-person correlation for unknown reasons
   - Occasions closer together may just be more related
   - More likely for outcomes that fluctuate (than change) over time

2-Level Models for the Variances

- **Where does the correlation or 'dependency' go?**
  Into a new random effects variance component ('pile of variance')



Summary: The Point of MLM

- All we've done so far is **carve up** our total variance into up to 3 piles:
  - BP (error) variance around intercept
  - BP (error) variance around slope
  - WP (error) residual variance

  } These two are one pile of "error variance" in RM Anova
- **But making piles does not make error variance go away…**

Level 1 (one source of) Within-Person Variation:

Gets accounted for by time-level predictors

Residual Variance (σ_e^2)

FIXED effects make variance go away (explain variance).

RANDOM effects just make a new pile of variance.

Level 2 (two sources of) Between-Person Variation:

Gets accounted for by person-level predictors

BP Int Variance (τ_{U0}^2)

BP Slope Variance (τ_{U1}^2)

τ_{U01} covariance

What can MLM do for you?

1. **Model dependency across observations**
    - Longitudinal, clustered, and/or cross-classified data? No problem!
    - Tailor your model of correlation over time and person to your data
2. **Include categorical or continuous predictors at any level**
    - Time-varying, person-level, group-level predictors for each variance
    - Explore reasons for dependency, don't just control for dependency
3. **Does not require same data structure for each person**
    - Unbalanced or missing data? No problem!
4. **You already know how (even if you don't know it yet)!**
    - SPSS Mixed, SAS Mixed, Stata, Mplus, R, HLM…
    - What's an intercept? What's a slope? What's a pile of variance?