Higher-Order Factor Models

• Topics:
  - The Big Picture
  - Identification of higher-order models
  - Measurement models for method effects
  - Equivalent models
Sequence of Steps in CFA or IFA

1. **Specify your measurement model(s)**
   - How many factors/thetas, which items load on which factors, and whether your need any method factors or error covariances
   - For models with large numbers of items, you should start by modeling each factor in its own analysis to make sure *each* factor fits its items

2. **Assess model fit, per factor, when possible (if 4+ indicators)**
   - **Global model fit**: Does a one-factor model adequately fit each set of indicators thought to measure the same latent construct?
   - **Local model fit**: Are any of the leftover covariances problematic? Any items not loading well (or are too redundant) that you might drop?
   - **Reliability/Info**: Are your standardized loadings practically meaningful?

3. **Once your single-factor measurement models are good, it’s time to consider the (higher-order) structural model**
Higher-Order Factor Models

• Purpose: What kind of higher-order factor structure best accounts for the covariance among the measurement model factors (not items)?
  ➢ In other words, what should the structural model among the factors look like?
  ➢ Best-fitting baseline for the structural model has all possible covariances among the lower-order measurement model factors → saturated structural model
  ➢ Just as the purpose of the measurement model factors is to predict covariance among the items, the purpose of the higher-order factors is to predict covariance among the measurement model factors themselves
  ➢ A single higher-order factor would be suggested by similar magnitude of correlations across the measurement model factors

• Note that distinctions between CFA, IFA, and other measurement models for different item types are no longer relevant at this point
  ➢ Factors and thetas are all multivariate normal latent variables, so a higher-order model is like a CFA regardless of the measurement model for the items
  ➢ Latent variables don’t have means apart from their items, so those are irrelevant
Necessary Measurement Model
Scaling to fit Higher-Order Factors

“Marker Item” for factor loadings
→ Fix 1 item loading to 1
→ **Estimate** factor variance

Because it will become “factor variance leftover” = “disturbance”, factor variance can’t be **fixed** (it must be estimated)

“Z-Score” for item intercepts or thresholds
→ Fix factor mean to 0
→ **Estimate** all intercepts/thresholds

All the factor means will be 0 and you generally won’t need to deal with them in the structural model anyway.
Identifying a 3-Factor Structural Model

**Option 1: 3 Correlated Factors**

**Measurement Model for Items:**
item variances, covariances, and means

Possible df = \((12*13) / 2 + 12 = 90\)
Estimated df = \(9\lambda + 12\mu + 12\sigma_e^2 = 33\)
df = 90 - 33 = 57 \(\rightarrow\) over-identified

**Structural Model for Factors:**
factor variances and covariances, no means

Possible df = \((3*4) / 2 + 0 = 6\)
Estimated df = 3 variances + 3 covariances
df = 6 - 6 = 0 \(\rightarrow\) just-identified

\[
\begin{align*}
\text{Var}(F_1) &= \kappa_1 = 0 \\
\text{Var}(F_2) &= \kappa_2 = 0 \\
\text{Var}(F_3) &= \kappa_3 = 0
\end{align*}
\]
Option 2a: 3 Factor “Indicators”
(Higher-Order Factor Variance = 1)

Same Measurement Model for Items:
Possible df = \((12 \times 13) / 2 + 12 = 90\)
Estimated df = \(9\lambda + 12\mu + 12\sigma_e^2 = 33\)
df = \(90 - 33 = 57\)
\(\rightarrow\) over-identified

New Structural Model for Factors:
Possible df = \((3 \times 4) / 2 + 0 = 6\)
Estimated df = \(3\lambda + 3\sigma_d^2\)
df = \(6 - 6 = 0\)
\(\rightarrow\) just-identified

Leftover factor variances (part of factor not predicted by higher-order factor) are called “disturbances”

If you only have 3 factors, both models will fit the same—the structural model is just-identified, and thus the fit of a higher-order factor CANNOT be tested
Option 2b: 3 Factor “Indicators”
(using Marker Lower-Order Factor)

Same Measurement Model for Items:
Possible df = \((12\times13) / 2 + 12 = 90\)
Estimated df = \(9\lambda + 12\mu + 12\sigma_e^2 = 33\)
df = 90 – 33 = 57
→ over-identified

New Structural Model for Factors:
Possible df = \((3\times4) / 2 + 0 = 6\)
Estimated df = \(2\lambda + 1\sigma_f^2 + 3\sigma_d^2\)
df = 6 – 6 = 0
→ just-identified

Leftover factor variances (part of factor not predicted by higher-order factor) are called “disturbances”

If you only have 3 factors, both models will fit the same—the structural model is just-identified, and thus the fit of a higher-order factor CANNOT be tested
**Structural Model Identification: 2 Factor “Indicators”**

**Measurement Model for Items:**
Possible df = \( \frac{(12 \times 13)}{2} + 12 = 90 \)
Estimated df = \( 8\lambda + 12\mu + 12\sigma^2_e = 32 \)
\( df = 90 - 32 = 58 \rightarrow \text{over-identified} \)

**Structural Model for Factors:**
Possible df = \( \frac{(4 \times 5)}{2} + 0 = 10 \)
Estimated df = \( 4\lambda + 0\sigma^2_F + 1\sigma_{F,F} + 4\sigma^2_d \)
\( \quad \text{— OR —} \)
Estimated df = \( 2\lambda + 2\sigma^2_F + 1\sigma_{F,F} + 4\sigma^2_d \)
\( df = 10 - 9 = 1 \rightarrow \text{over-identified} \)

However, this model will not be identified structurally unless there is a non-0 covariance between the higher-order factors.
Higher-Order Factor Identification

• Possible structural df depends on # of measurement model factor variances and covariances (NOT # items)

  ➢ 2 measurement model factors $\rightarrow$ Under-identified
    ▪ They can be correlated, which would be just-identified... that’s it

  ➢ 3 measurement model factors $\rightarrow$ Just-identified
    ▪ They can all be correlated OR a single higher-order factor can be fit
    ▪ Some # variance/disturbances per factor (so, 3 total) in either option
    ▪ Factor variances and covariances will be perfectly reproduced

  ➢ 4 measurement model factors $\rightarrow$ Can be over-identified
    ▪ They can all be correlated (6 correlations required; just-identified)
    ▪ They can have a higher-order factor (4 loadings; over-identified)
    ▪ The fit of the higher-order factor can now be tested
Examples of Structural Model Hypothesis Testing

• Do I have a higher-order factor of my subscale factors?
  ➢ If 4 or more subscale factors: Compare fit of alternative models
    ▪ Saturated Baseline: All 6 factor covariances estimated freely
      Alternative: 1 higher-order factor instead (so df=2)—is model fit WORSE?
  ➢ If 3 (or fewer) subscale factors: CANNOT BE DETERMINED
    ▪ Saturated baseline and alternative models will fit equivalently

• Do I need a residual covariance, but I’m doing IFA in ML?
  ➢ Predict those two items with a factor, fix both loadings=1 if you need a positive covariance or $-1/+1$ if you need a negative covariance
  ➢ Estimate its factor variance, which then becomes the residual covariance

• Do I have need additional “method factors”?
  ➢ Some examples...
Maydeu-Olivares & Coffman (Psychological Methods, 2006) present 4 models by which to measure a latent factor of optimism using the 3 positively and 4 negatively worded items shown below

A: Single factor (df = 14)

B: Two wording factors (df = 13)

C: Three-factor “Bifactor” model (df = 7)

D: “Random Intercept” 2-factor model (df = 13)
What to do with method effects?

Maydeu-Olivares & Coffman (2006) present 4 ways to measure a latent factor of optimism with 3 positively and 4 negatively worded items.

A: Single “optimism” factor (which doesn’t fit well)

\[
\text{Opt BY } \text{i1* i4* i5*;}
\]
\[
\text{i3* i8* i9* i12*;}
\]
\[
\text{Opt@1; [Opt@0];}
\]

B: “Optimism” and “Pessimism” two-factor model (fits better)

\[
\text{Opt BY } \text{i1* i4* i5*;}
\]
\[
\text{PES BY i3* i8* i9* i12*;}
\]
\[
\text{Opt WITH Pes*;}
\]
\[
\text{Opt@1; [Opt@0];}
\]
\[
\text{Pes@1; [Pes@0];}
\]
One- vs. Two-Factor Models

Negatively-worded items 3, 8, 9, and 12 were not reverse-coded.

Without recoding, factor covariance would be negative.

Note: a higher-order factor could be included if both loadings were fixed to 1, but it would fit the same as just allowing the two wording factors to covary.

<table>
<thead>
<tr>
<th>Item</th>
<th>Optimism</th>
<th>Two-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Optimism</td>
<td>Optimism</td>
</tr>
<tr>
<td>Item 1</td>
<td>0.38</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Item 2</td>
<td>0.48</td>
<td>0.78</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Item 3</td>
<td>0.46</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Item 4</td>
<td>-0.64</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Item 5</td>
<td>-0.86</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Item 6</td>
<td>-0.79</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td></td>
</tr>
<tr>
<td>Item 7</td>
<td>-0.70</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td></td>
</tr>
</tbody>
</table>
Bifactor Model Fits Well...

General “optimism” factor is measured by all items

Specific factors are measured only by items with that type of wording and are both uncorrelated

2 problems in interpreting these factors as desired:
1) “Specific” positive loadings > “general” loadings
2) Specific negative loadings are weak or non-significant (indicating model is over-parameterized)
Random Intercept Factor Fits Well…

- General “optimism” factor is measured by all items (all loadings estimated)
- New “random intercept” factor allows for constant person shifts across items (e.g., due to different response scale interpretations); Variance = 0.13 here

Opt BY i1* i4* i5* 
  i3* i8* i9* i12*;
Opt@1; [Opt@0];
Int BY i1@1 i4@1 i5@1 
  i3@1 i8@1 i9@1 i12@1;
Int*; [Int@0];
Opt WITH Int@0;

<table>
<thead>
<tr>
<th>One-factor random intercept: Optimism</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.54 (0.05)</td>
</tr>
<tr>
<td>0.66 (0.05)</td>
</tr>
<tr>
<td>0.61 (0.05)</td>
</tr>
<tr>
<td>-0.56 (0.05)</td>
</tr>
<tr>
<td>-0.78 (0.05)</td>
</tr>
<tr>
<td>-0.71 (0.05)</td>
</tr>
<tr>
<td>-0.65 (0.05)</td>
</tr>
</tbody>
</table>
Heartland Forgiveness Scale (HFS)


Model 4. Six correlated lower-order factors for positive and negative self, other, and situation “forgiveness” and “not unforgiveness” (reverse-coded)

**Total possible df for 18 items = 189**

\[
\frac{v \times (v + 1)}{2} + v = \frac{18 \times 19}{2} + 18 = 189
\]

**Measurement Model = 48 parameters**

\[12\lambda + 18\mu + 18\sigma_e^2\]

**Structural Model = 21 parameters**

\[6\sigma_F^2, \text{ 15 factor covariances (all possible)}\]

**Total model df = 189 – 69 = 120**
Model 5. Six lower-order factors for positive and negative self, other, and situation forgiveness and not unforgiveness as before, but now 3 higher-order correlated factors of Self, Other, and Situation, and 2 uncorrelated wording factors.

Structural Model = 8 parms
(DF = 21 - 8 = 13)

! Constant Method Effects
Pos BY SelfPos* (5)
OtherPos* (5)
SitPos* (5);
Neg BY SelfNeg* (5)
OtherNeg* (5)
SitNeg* (5);

! No method factor cov.
Self@1 Other@1 Sit@1;
Self WITH Other* Sit*;
Other WITH Sit*;
Pos@1 Neg@1; Pos WITH Neg@0;
Pos Neg WITH Self@0 Other@0 Sit@0;

! Constant factor disturbances
SelfPos* OtherPos* SitPos* (3);
SelfNeg* OtherNeg* SitNeg* (4);
Equivalency across Models

- Remember, the purpose of a measurement model is to reproduce the observed covariance matrix and means of the items
- This means that models that generate the same predicted covariance matrix and means are equivalent models
- This will often not be comforting, but it is the truth...
- Here’s an example: These models make very different theoretical statements, but they will nevertheless fit equivalently

\[
\begin{align*}
A & \rightarrow B & .60 & \rightarrow C & .50 \\
& & e_B & .64 & \\
& & e_C & .75 & \\
B & \rightarrow C & .50 & \\
C & \rightarrow B & .60 & \\
& & e_B & .75 & \\
& & e_A & .64 & \\
\end{align*}
\]

- Generally speaking, the fewer df left over (i.e., the more complicated the model), the more equivalent alternative solutions there are
More Equivalent Models…

Top: One can think these 4 items as “effects” (indicators) of depression...

Left: One can think of any one item as “causing” depression and the others as “effects” of depression...

**Point of the story: CFA/SEM cannot give you TRUTH.** Contrary to what it’s often called, SEM is not really “causal” modeling.
Wrapping Up…

- Fitting measurement and structural models are two separate issues:
  - **Measurement model**: Do my lower-order factors account for the observed covariances among my ITEMS?
  - **Structural model**: Do higher-order factors account for the estimated covariances among my measurement model FACTORS/THETAS?
    - A higher-order factor is NOT the same thing as a ‘total score’ though

- Figure out the measurement models FIRST, then structural models
  - Recommend fitting measurement models separately per factor, then bringing them together once you have each factor/theta settled
  - This will help to pinpoint the source of misfit in complex models

- Keep in mind that structural models may not be ‘unique’
  - Mathematically equivalent models can make very different theoretical statements, so there’s no real way to choose between them if so…